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TILINGS OF SIMPLEXES, SCHEMES OF GLUED GRAPHS AND COMPACTIFICATION OF $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$

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Introduction

For all integers $r \geq 2$ and $n \geq 1$, we construct a projective compactification $\Omega^{r,n}$ of $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$ which satisfies the following properties: it is equipped with an action of PGL_r^{n+1}, it is normal and even toroidal (more precisely, it has a smooth morphism on the quotient stack of a certain toric variety $\mathcal{A}^{r,n}$ by its torus $\mathcal{A}_\emptyset^{r,n}$), it is naturally stratified as disjoint union of locally closed subschemes $\overline{\Omega}_{\underline{S}}^{r,n}$ indexed by certain tilings $\underline{S}$ of simplex $S^{r,n}$ of dimension n and of side r and which are smooth and admitting a modular description.

There are several possible starting points.

One is the construction by De Concini and Procesi of compactification of $(G \times G)/G$ for any adjoint semisimple group G ([DP83]). For this purpose, they take representations V of $G \times G$ in which exists a line h whose stabilizer is G diagonally embedded in $G \times G$; then the schematic closure in $\mathbb{P}(V)$ of the orbit of point h is a equivariant projective compactification of $(G \times G)/G$.

In our situation, we consider the representation of PGL_r^{n+1} obtained in the following way: denote $\mathbb{A}^r$ the affine space of dimension r, we make GL_r^{n+1} act on $(\mathbb{A}^r)^{n+1}$ and thus on $\Lambda^r(\mathbb{A}^r)^{n+1}$ then, $S^{r,n}$ denote the simplex

$$\{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1} \mid i_0 + i_1 + \cdots + i_n = r\},$$

we decompose $\Lambda^r(\mathbb{A}^r)^{n+1}$ into

$$\bigoplus_{(i_0,\ldots,i_n)\in S^{r,n}} \Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r$$

and we see that PGL_r^{n+1} act on

$$\prod_{(i_0,\ldots,i_n)\in S^{r,n}} \mathbb{P}(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r).$$

If $\mathbb{A}^r$ is diagonally embedded in $\mathbb{A}_r^{n+1}$, $\Lambda^r\mathbb{A}^r$ is identified with a line of $\Lambda^r\mathbb{A}_r^{n+1}$ whose stabilizer in GL_r^{n+1} is GL_r diagonally embedded; in addition, $\Lambda^r\mathbb{A}^r$ is projected on a line in each of the factors

$$\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r$$

of $\Lambda^r\mathbb{A}_r^{n+1}$ and the stabilizer of induced point in

$$\prod_{(i_0,\ldots,i_n)\in S^{r,n}} \mathbb{P}(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r)$$

is PGL_r diagonally embedded in PGL_r^{n+1}. Then the orbit of this point is identified with $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$ and the equivariant compactification $\overline{\Omega}^{r,n}$ is obtained from the schematic closure of this one in

$$\prod_{(i_0,\ldots,i_n)\in S^{r,n}} \mathbb{P}(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r)$$

by a few simple blow-ups (intended to separate the strata). When $n = 1$, $\overline{\Omega}^{r,1}$ is none other than the compactification of De Concini and Procesi of

$$(\mathrm{PGL}_r \times \mathrm{PGL}_r)/\mathrm{PGL}_r\,.$$

Another starting point is the stratification of the Grassmannians into "thin Schubert cells" (see for example [Gel+87]). We introduce the Grassmannian $\mathrm{Gr}^{r,n}$ of subspaces of dimension r in $(\mathbb{A}^r)^{n+1}$; this is a closed subscheme of $\mathbb{P}\big(\Lambda^r(\mathbb{A}^r)^{n+1}\big)$. For any point F of $\mathrm{Gr}^{r,n}$ represented by a nonzero element u of

$$\Lambda^r(\mathbb{A}^r)^{n+1} = \bigoplus_{(i_0,\ldots,i_n)\in S^{r,n}} \Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r$$

of which we denote $u_{i_0,\ldots,i_n}$ the components of it, the subset

$$S = \{(i_0,\ldots,i_n)\in S^{r,n} \mid u_{i_0,\ldots,i_n} \neq 0\}$$

is an entier convex polytope in the sense that there is a family of integers d_I^S indexed by subsets I of $\{0,\ldots,n\}$ satisfying

$$d_\emptyset^S = 0,\ d^S_{\{0,\ldots,n\}} = r,$$

$$d_I^S + d_J^S \leq d^S_{I\cup J} + d^S_{I\cap J},\ \forall\, I, J,$$

and

$$S = \{(i_0,\ldots,i_n)\in S^{r,n} \mid \sum_{\alpha\in I} i_\alpha \geq d_I^S,\ \forall\, I\}.$$

And if $E_0,\ldots,E_n$ denotes the $n+1$ factors $\mathbb{A}^r$ of $E = (\mathbb{A}^r)^{n+1}$ and E_I, $I \subsetneq \{0,\ldots,n\}$, denotes

$$\bigoplus_{\alpha\in I} E_\alpha,$$

this family (d_I^S) is given by $d_I^S = \dim(F\cap E_I)$, $\forall\, I$. Thus, for such an entire convex polyhedron S, the scheme $\mathrm{Gr}_S^{r,n}$ of tuples in

$$\mathbb{G}_m\backslash \prod_{(i_0,\ldots,i_n)\in S} \Big[(\Lambda^{i_0}\mathbb{A}^r\otimes\cdots\otimes\Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

which, completed by 0 in indices outside S, are in $\mathrm{Gr}^{r,n}$, classifies the subspaces F of E such that

$$\dim(F\cap E_I) = d_I^S,\ \forall\, I.$$

When S describes(it seems to mean that S to be any element of this set) the set of all entire convex polyhedrons of $S^{r,n}$, $\mathrm{Gr}_S^{r,n}$ constitute a stratification of $\mathrm{Gr}^{r,n}$.

We mark equally that any "boundary" S' of an entire convex polyhedron S defined by an equation of the form

$$\sum_{\alpha\in I} i_\alpha = d_I^S$$

is an entier convex polytope and that the morphism of restriction to this "boundary"

$$(u_{i_0,\ldots,i_n})_{(i_0,\ldots,i_n)\in S} \mapsto (u_{i_0,\ldots,i_n})_{(i_0,\ldots,i_n)\in S'}$$

sends $\mathrm{Gr}_S^{r,n}$ to $Gr_{S'}^{r,n}$ and represents

$$F \mapsto (F\cap E_I)\oplus F/(F\cap E_I) \subsetneq E_I \oplus E/E_I = E.$$

In consequence, for any entire tiling (i.e. whose tiles are entire convex polyhedron) $\underline{S}$ of simplex $S^{r,n}$, we can glue schemes $\mathrm{Gr}_S^{r,n}$ associated with tiles S of $\underline{S}$ of along schemes $\mathrm{Gr}_{S'}^{r,n}$ associated with "boundaries" S' shared by two tiles of $\underline{S}$: no introduce the closed subscheme $\mathrm{Gr}_{\underline{S}}^{r,n}$ of

$$\mathbb{G}_m\backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r\otimes\cdots\otimes\Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

of tuples $(u_s)_{s\in S^{r,n}}$ such that, for any tile S of $\underline{S}$, the restraint tuple $(u_s)_{s\in S}$ is in $\mathrm{Gr}_S^{r,n}$. They classify the families (F_S) of subspaces of dimension r in E indexed by tile S of $\underline{S}$ such that:

– for any tile S of $\underline{S}$ and any subset I of $\{0,\ldots,n\}$,

$$\dim(F_S\cap E_I)=d_I^S,$$

– for any tiles S, S' of $\underline{S}$ having in common a "boundary" of equation

$$\sum_{\alpha\in I} i_\alpha = d_I^S = r - d_J^{S'}$$

where $J=\{0,\ldots,n\}-I$,

$$F_S\cap E_I = F_{S'}/(F_{S'}\cap E_J),\quad F_S/(F_S\cap E_I)=F_{S'}\cap E_J.$$

In particular, we show that if $\underline{S}$ is a "convex" entire tiling, the glued graph scheme $\mathrm{Gr}_{\underline{S}}^{r,n}$ is smooth of dimension nr^2 thus independent of $\underline{S}$.

We construct on the other hand a (normal) toric variety $\mathcal{A}^{r,n}$ of torus

$$\mathcal{A}_\emptyset^{r,n}=\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m^{n+1}$$

whose orbits $\mathcal{A}_{\underline{S}}^{r,n}$ correspond exactly to convex entire tilings $\underline{S}$ of simplex $S^{r,n}$, and in such a way that an orbit $\mathcal{A}_{\underline{S}'}^{r,n}$ is contained in the closure of another $\mathcal{A}_{\underline{S}}^{r,n}$ if and only if the tiling $\underline{S}'$ refines the tiling $\underline{S}$. Then we build a quasi-projective scheme $\Omega^{r,n}$ equipped with an action of

$$\mathrm{GL}_r^{n+1}\times\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$$

and with a morphism

$$\Omega^{r,n}\to\mathcal{A}^{r,n}$$

equivariant under $\mathbb{G}_m^{S^{r,n}}$; this morphism is smooth of relative dimension nr^2 and its fiber above marked point $\alpha_{\underline{S}}$ of any orbit $\mathcal{A}_{\underline{S}}^{r,n}$ in $\mathcal{A}^{r,n}$ is identified with glued graph scheme $\mathrm{Gr}_{\underline{S}}^{r,n}$. Then $\Omega^{r,n}$ is stratified by the preimage of $\Omega_{\underline{S}}^{r,n}$ of orbits $\mathcal{A}_{\underline{S}}^{r,n}$ in $\mathcal{A}^{r,n}$ and each one identifies with

$$\mathbb{G}_m^{\underline{S}}\backslash(\mathbb{G}_m^{S^{r,n}}\times\mathrm{Gr}_{\underline{S}}^{r,n})$$

where $\mathbb{G}_m^{\underline{S}}$ denotes the sub-torus of $\mathbb{G}_m^{S^{r,n}}$ stabilizer of the marked point $\alpha_{\underline{S}}$. Finally, the compactification $\overline{\Omega}^{r,n}$ of $\mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r$ is obtained as quotient of $\Omega^{r,n}$ by the action of $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$ which is free; its strata $\overline{\Omega}_{\underline{S}}^{r,n}$ are the quotients of $\Omega_{\underline{S}}^{r,n}$ by this same action and they are also identified with the quotients $(\mathbb{G}_m^{\underline{S}}/\mathbb{G}_m)\backslash\mathrm{Gr}_{\underline{S}}^{r,n}$.

Let's say again in this introduction that for any map

$$\{0,\ldots,p\}\to\{0,\ldots,n\},$$

the induced morphism

$$\mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r\to\mathrm{PGL}_r^{p+1}/\mathrm{PGL}_r$$

is extended to a morphism

$$\overline{\Omega}^{r,n}\to\overline{\Omega}^{r,p}.$$

The family of $\overline{\Omega}^{r,n}$, $n\geq 1$, equipped with these induced morphisms constitutes a simplicial scheme which extends the simplicial classifying scheme of PGL_r formed of $\mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r$.

If the base field is finite with q elements and therefore we have the Frobenius endomorphisms τ of elevation to the power q, the existence of morphisms

$$p_0, p_1, p_2 : \overline{\Omega}^{r,2} \to \overline{\Omega}^{r,1}$$

deduced from the three ascending injections

$$\{0,1\} \to \{0,1,2\}$$

allows to build a compactification of the Lang isogeny of PGL_r i.e. an equivariant projective compactification $\overline{\Omega}^{r,\tau}$ of PGL_r equipped with a morphism

$$\overline{\Omega}^{r,\tau} \to \overline{\Omega}^{r,1}$$

which extends

$$\begin{aligned} \mathrm{PGL}_r &\to \mathrm{PGL}_r \\ g\tau(g)^{-1} &\circ g. \end{aligned}$$

The scheme $\overline{\Omega}^{r,\tau}$ is also toroidal (and even it has a smooth morphism on the quotient toric stack of a certain normal toric variety $\mathcal{A}^{r,\tau}$ by its torus $\mathcal{A}^{r,\tau}_{\emptyset}$). In the category of schemes equipped with a morphism on a toric stack,

$$(\overline{\Omega}^{r,\tau} \to \mathcal{A}^{r,\tau}/\mathcal{A}^{r,\tau}_{\emptyset})$$

is defined as the kernel of the diagram

$$(\overline{\Omega}^{r,2} \to \mathcal{A}^{r,2}/\mathcal{A}^{r,2}_{\emptyset}) \underset{\tau\circ p_1}{\overset{p_0}{\rightrightarrows}} (\overline{\Omega}^{r,1} \to \mathcal{A}^{r,1}/\mathcal{A}^{r,1}_{\emptyset})$$

and $\overline{\Omega}^{r,\tau} \to \overline{\Omega}^{r,1}$ is then induced by $p_2 : \overline{\Omega}^{r,2} \to \overline{\Omega}^{r,1}$. The scheme $\overline{\Omega}^{r,\tau}$ is disjoint union of locally closed subschemes $\overline{\Omega}^{r,\tau}_{\underline{S}}$ indexed by certain convex entire tiling $\underline{S}$ of triangle $S^{r,2}$; these strata are smooth and admit a modular description.

The construction explained in this article is transportable to many other situations: here, we consider a space E with a graduation whose factors $E_0, \dots, E_n$ have the same dimension, but we could as well take factors of different dimensions or a filtration instead of a graduation... We will explain in the next article how such a variant allows us to construct compactifications of G^{n+1}/G for G a parabolic subgroup of PGL_r and then for G a transform of PGL_r by Weil restriction of scalars, with an immediate map to the compactification of the Lang isogeny on such a transform; these compactifications are smooth over the quotient field of a certain toric variety by its torus and they are disjoint unions of locally closed smooth and modular strata. In particular, we obtain compactifications of the stacks of Drinfeld shtuka with arbitrary level structures by forming simple fiber bundles. The particular case of level structures without multiplicities for which the compactification of $\mathrm{PGL}_r^3 / \mathrm{PGL}_r$ and its map to that of the Lang isogeny in PGL_r was announced in a note to the Proceedings of the Academy of Sciences (series I, volume 325, pg 1309-1312,1997) and detailed in a preprint from Orsay. Note that Faltings also mentions the quotients G^{n+1}/G, for G a reductive group, in relation with the study of singularities of local models (see[Fal97][Concluding remarks]). Moreover, when G is a classical group, it is not difficult to adapt our work to build compactifications of G^{n+1}/G satisfying always the same type of properties.

The present paper is organized as follows: The actual construction with its main properties is given in Chapter 1 and the proofs related to the toric variety $\mathcal{A}^{r,n}$, to the schemes of glued graphs $\mathrm{Gr}^{r,n}_{\underline{S}}$ and to the global schemes $\Omega^{r,n}$ and $\overline{\Omega}^{r,n}$ are collected in Chapter 2, 3 and 4 respectively; the map to Lang's isogeny is given in the last Section 4.4.

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Chapter 1

Definition, description and statement of main properties

1.1 The scheme of convex entire tilings of simplex

In this first Section, we will construct a toric variety whose set of orbits is naturally identified with the set of "entire convex tilings" of the simplex. First we want to define these tilings.

Let $r \geq 2$ and $n \geq 1$ two integers.

Let

$$\mathbb{R}^{r,n} = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + x_1 + \cdots + x_n = r\}$$

which is a real affine space of dimension n. It contains the lattice

$$\mathbb{Z}^{r,n} = \{(i_0, \ldots, i_n) \in \mathbb{Z}^{n+1} \mid i_0 + i_1 + \cdots + i_n = r\}$$

that we can call a lattice of integer points of $\mathbb{R}^{r,n}$. And let

$$S^{r,n} = \{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1} \mid i_0 + i_1 + \cdots + i_n = r\}$$

which is the set of integer points of simplex

$$\overline{S^{r,n}} = \{(x_0, \ldots, x_n) \in \mathbb{R}_+^{n+1} \mid x_0 + x_1 + \cdots + x_n = r\}$$

in $\mathbb{R}^{r,n}$.

In the real affine space $\mathbb{R}^{r,n}$ of dimension n, we call a *"convex polyhedron"* to be any convex subset generated by a finite number of points. We can define the dimension of such a polyhedron as well as its faces which are also convex polyhedra and in particular its boundaries i.e. its faces of codimension 1. We also call *"(convex) tiles"* to be those convex polyhedra which are of maximal dimension n. Finally, a *"tiling"* of a certain tile is a writing of this tile as a union of smaller tiles whose interiors do not meet.

Definition 1.1. A convex polyhedron S of the space $\mathbb{R}^{r,n}$ will be said to be *"entire"* (i.e. matroid) if it is of the form

$$S = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{0 \leq \alpha \leq n} x_\alpha = r \text{ and } \sum_{\alpha \in I} x_\alpha \geq d_I, \ \forall \ I\}$$

for (d_I) a family of integers of $\mathbb{Z}$ indexed by the subsets I of $\{0, \ldots, n\}$ which is *convex in the sense that*

$$d_\emptyset = 0, \ d_{\{0,\ldots,n\}} = r, \ d_I + d_J \leq d_{I\cap J} + d_{I\cup J}, \forall I, J.$$

Note that the term entire convex polyhedron has for us a more restrictive meaning than usually. We will demonstrate:

Lemma 1.2. (i) *If S is a entire convex polyhedron defined by a convex family of integers (d_I) as in the definition 1.1, we have for any subset $I \subseteq \{0, \ldots, n\}$*

$$d_I = \min_{(x_0,\ldots,x_n)\in S} \{\sum_{\alpha \in I} x_\alpha\}$$

so that the family (d_I) is uniquely determined by S.

(ii) *The faces of an entire convex polyhedron are also entire convex polyhedra.*

(iii) *Any entire convex polyhedron is generated by its integer points so that it can be identified with the finite subset of these.*

(iv) *If we call "generating family" to be any family of $n+1$ integer points that generates the lattice of integer points, any entire tile contains at least one "generating family".*

(v) *A tile which admits a tiling constituted of entire tiles is itself entire.*

Of course, we will call an "entire tiling" of an entire tile to be a tiling of the tile which all the tiles are themselves entire.

In the real vector space of finite dimension of functions $S^{r,n} \to \mathbb{R}$, let $\mathcal{C}^{r,n}$ be the cone of functions $v : S^{r,n} \to \mathbb{R}$ such that for any affine map $l : S^{r,n} \to \mathbb{R}$ satisfying $l \leq v$, the set

$$\{s \in S^{r,n} \mid l(s) = v(s)\}$$

is an entire convex polyhedron.

And if $\underline{S}$ is an entire tiling of $S^{r,n}$, let $\mathcal{C}^{r,n}_{\underline{S}}$ be the convex cone of functions $v : S^{r,n} \to \mathbb{R}$ such that for any tile S of $\underline{S}$ there exists an affine map $l_S : S^{r,n} \to \mathbb{R}$ satisfying $l_S \leq v$ and

$$S = \{s \in S^{r,n} \mid l_S(s) = v(s)\}.$$

Those of the entire tilings $\underline{S}$ of $S^{r,n}$ for which $\mathcal{C}^{r,n}_{\underline{S}}$ is not empty will be called the "convex entire tilings" of $S^{r,n}$. We remark that if $\emptyset$ denotes the trivial tiling of $S^{r,n}$, then $\mathcal{C}^{r,n}_{\emptyset}$ is the subspace of affine functions $S^{r,n} \to \mathbb{R}$.

We will prove:

Proposition 1.3. (i) *The cone $\mathcal{C}^{r,n}$ is the disjoint union of convex cones $\mathcal{C}^{r,n}_{\underline{S}}$ with $\underline{S}$ describing the set of convex entire tilings of $S^{r,n}$.*

(ii) *For any $\underline{S}$, the closure $\overline{\mathcal{C}^{r,n}_{\underline{S}}}$ of $\mathcal{C}^{r,n}$ is the disjoint union of $\mathcal{C}^{r,n}_{\underline{S}'}$ with $\underline{S}'$ describing the set of convex entire tilings of $S^{r,n}$ more coarse than $\underline{S}$.*

In addition, $\overline{\mathcal{C}^{r,n}_{\underline{S}}}$ is a rational polyhedral convex cone (i.e. generated by a finite number of its elements taking their values in $\mathbb{Z}$) and faces of $\overline{\mathcal{C}^{r,n}_{\underline{S}}}$ are $\overline{\mathcal{C}^{r,n}_{\underline{S}'}}$ with $\underline{S}'$ more coarse than $\underline{S}$.

(iii) *For $\underline{S}$ and $\underline{S}'$ two convex entire tilings of $S^{r,n}$, the set of convex entire tilings of $S^{r,n}$ more coarse than both $\underline{S}$ and $\underline{S}'$ has a smallest element $\underline{S} \vee \underline{S}'$. And the intersection of $\overline{\mathcal{C}^{r,n}_{\underline{S}}}$ and $\overline{\mathcal{C}^{r,n}_{\underline{S}'}}$ is equal to $\overline{\mathcal{C}^{r,n}_{\underline{S} \vee \underline{S}'}}$.*

Let us note that in their study of Gelfand discriminants, Kapranov and Zelevinsky were also led to introduce certain cones of piecewise affine convex functions on polyhedra, with the associated polyhedral decompositions (see [Loe91][Section 1])

According to Proposition 1.3, the rational polyhedral convex cones $\overline{\mathcal{C}^{r,n}_{\underline{S}}}/\mathcal{C}^{r,n}_{\emptyset}$ form a fan in the quotient of the space of functions $S^{r,n} \to \mathbb{R}$ by the subspaces of affine functions. The general theory of toric variety such as shown in [Kem+73] associates to this fan a normal toric variety $\mathcal{A}^{r,n}$ of torus $\mathcal{A}^{r,n}_{\emptyset} = \mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m^{n+1}$ where the torus $\mathbb{G}_m^{n+1}$ is embedded in the torus $\mathbb{G}_m^{S^{r,n}}$ by

$$(\lambda_0, \dots, \lambda_n) \mapsto (\lambda_0 \lambda_1^{-i_1} \cdots \lambda_n^{-i_n})_{(i_0,\dots,i_n) \in S^{r,n}}.$$

The orbits in $\mathcal{A}^{r,n}$ are locally closed subschemes indexed naturally by the convex entire tilings $\underline{S}$ of $S^{r,n}$; we denote $\mathcal{A}_{\underline{S}}$. Each has a marked point $\alpha_{\underline{S}}$. The closure of an orbit $\mathcal{A}_{\underline{S}}$ is the union of $\mathcal{A}_{\underline{S}'}$ for $\underline{S}'$ refining $\underline{S}$, and union of $\mathcal{A}_{\underline{S}'}$ for $\underline{S}'$ coarser than $\underline{S}$ is the smallest invariant open subset containing $\mathcal{A}_{\underline{S}}$.

We can therefore call $\mathcal{A}^{r,n}$ the *"scheme of convex entire tilings"* of $S^{r,n}$. When $n=1$, the convex entire tilings of $S^{r,1}$ are the partitions of the interval $\{0,\dots,r\}$ and the toric variety $\mathcal{A}^{r,1}$ of torus $\mathbb{G}_m^{r+1}/\mathbb{G}_m^2 \cong \mathbb{G}_m^{r-1}$ is identified with the affine space $\mathbb{A}^{r-1}$ of dimension $r-1$. When $n=2$, the convex entire tiling of $S^{r,2}$ constitutes of minimal equilateral triangles is finer than all the others so that the scheme $\mathcal{A}^{r,2}$ is affine; it is smooth if $r=2$ but it is not smooth if $r \geq 3$. When $n \geq 3$ finally, there are more than one minimal convex entire tilings of $S^{r,n}$ and the scheme $\mathcal{A}^{r,n}$ is not affine; it is smooth if $r=2$ but it is not smooth if $r \geq 3$.

1.2 Construction by schematic closure

Let's make the product GL_r^{n+1} of $n+1$ copies of linear group GL_r of rank r act on the sum $(\mathbb{A}^r)^{n+1}$ of $n+1$ copies of vector spaces $\mathbb{A}^r$ of dimension r and thus also on the exterior power $\Lambda^r(\mathbb{A}^r)^{n+1}$. If $\mathbb{A}^r$ is diagonally embedded in $(\mathbb{A}^r)^{n+1}$, $\Lambda^r\mathbb{A}^r$ is identified with a line of $\Lambda^r(\mathbb{A}^r)^{n+1}$ whose stabilizer in GL_r^{n+1} is none other than GL_r diagonally embedded. This determines a locally closed immersion

$$\mathrm{GL}_r^{n+1} / \mathrm{GL}_r \hookrightarrow \mathbb{P}\big(\Lambda^r(\mathbb{A}^r)^{n+1}\big).$$

Its image has a closure the Grassmannian $\mathrm{Gr}^{r,n}$ classifying the subspaces of dimension r in $(\mathbb{A}^r)^{n+1}$; it is identified with the open subset of subspaces whose projections on each of $n+1$ factors $\mathbb{A}^r$ is isomorphism.

If we write the decomposition

$$\Lambda^r(\mathbb{A}^r)^{n+1} = \bigoplus_{(i_0,\dots,i_n)\in S^{r,n}} \Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r$$

where

$$S^{r,n} = \{(i_0,\dots,i_n) \in \mathbb{N}^{n+1} \mid i_0+\cdots+i_n = r\},$$

this image is also the open subset of subspaces of which none of the components in the factors

$$\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r$$

vanish. So we have a morphism

$$\mathrm{GL}_r^{n+1} / \mathrm{GL}_r \to \prod_{(i_0,\dots,i_n)\in S^{r,n}} \mathbb{P}(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r)$$

which is factorized through a locally closed immersion

$$\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r \hookrightarrow \prod_{(i_0,\dots,i_n)\in S^{r,n}} \mathbb{P}(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r).$$

Embed $\mathrm{GL}_r \times \mathbb{G}_m^{n+1}$ in $\mathrm{GL}_r^{n+1} \times \mathbb{G}_m^{S^{r,n}}$ by

$$(g;\lambda_0,\dots,\lambda_n) \mapsto \Big(g;\lambda_1 g,\dots,\lambda_n g; \big(\lambda_0(\det g)^{-1}\lambda_1^{-i_1}\cdots\lambda_n^{-i_n}\big)_{(i_0,\dots,i_n)\in S^{r,n}}\Big),$$

this immersion also appears as the quotient by the torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$ of the locally closed immersion

$$(\mathrm{GL}_r^{n+1} \times \mathbb{G}_m^{S^{r,n}})/(\mathrm{GL}_r \times \mathbb{G}_m^{n+1}) \hookrightarrow \mathbb{G}_m \backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

$$\Big(g_0,\dots,g_n;(\lambda_{i_0,\dots,i_n})\Big) \mapsto \Big(\lambda_{i_0,\dots,i_n}\cdot \Lambda^{i_0}({}^t g_0) \vee \cdots \vee \Lambda^{i_n}({}^t g_n)\Big)_{(i_0,\dots,i_n)\in S^{r,n}}$$

where

$$\lambda_{i_0,\ldots,i_n} \cdot \Lambda^{i_0}({}^t g_0) \vee \cdots \vee \Lambda^{i_n}({}^t g_n)$$

are homomorphisms

$$\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r \to \Lambda^{i_0}\mathbb{A}^r \wedge \cdots \wedge \Lambda^{i_n}\mathbb{A}^r = \Lambda^r\mathbb{A}^r = \mathbb{A}^1$$

identified with elements of $\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r$ by means of the canonical base of $\mathbb{A}^r$.

On the other hand, we have the composite morphism

$$(\mathrm{GL}_r^{n+1} \times \mathbb{G}_m^{S^{r,n}})/(\mathrm{GL}_r \times \mathbb{G}_m^{n+1}) \to \mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m^{n+1} = \mathcal{A}_\emptyset^{r,n} \hookrightarrow \mathcal{A}^{r,n}$$

with values in the scheme $\mathcal{A}^{r,n}$ of convex entire tilings of simplex $S^{r,n}$.

By making the product, we obtain a locally closed immersion

$$(\mathrm{GL}_r^{n+1} \times \mathbb{G}_m^{S^{r,n}})/(\mathrm{GL}_r \times \mathbb{G}_m^{n+1}) \hookrightarrow \mathbb{G}_m \backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big] \otimes \mathcal{A}^{r,n}.$$

We can now ask:

Definition 1.4. Let $\Omega^{r,n}$ be the schematic closure of the image $\Omega_\emptyset^{r,n}$ of

$$(\mathrm{GL}_r^{n+1} \times \mathbb{G}_m^{S^{r,n}})/(\mathrm{GL}_r \times \mathbb{G}_m^{n+1})$$

in

$$\mathbb{G}_m \backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big] \otimes \mathcal{A}^{r,n}.$$

And let $\overline{\Omega}^{r,n}$ be the quotient scheme of $\Omega^{r,n}$ by the (free) action of torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$.

Of course, $\Omega^{r,n}$ is a torsor on $\overline{\Omega}^{r,n}$ for the torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$. And on the other hand, $\overline{\Omega}^{r,n}$ contains as open dense subset the quotient $\overline{\Omega}_\emptyset^{r,n} \cong \mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r$ of $\Omega_\emptyset^{r,n}$ by the free of action of torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$.

We will demonstrate:

Theorem 1.5. *The morphism*

$$\Omega^{r,n} \to \mathbb{G}_m \backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

and its quotient by the free actions of torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$

$$\overline{\Omega}^{r,n} \to \mathbb{G}_m \backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \mathbb{P}(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r)$$

are projective.

In consequence, $\overline{\Omega}^{r,n}$ *realizes a projective compactification of* $\mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r$.

When $n = 1$, the compactification $\overline{\Omega}^{r,1}$ of $\mathrm{PGL}_r^2/\mathrm{PGL}_r$ is a particular case of the compactification of $(G \times G)/G$ constructed by De Concini and Procesi for any adjoint semisimple group G. On the other hand, $\Omega^{r,1}$ is classically known as the scheme for "complete homomorphisms" (see for example [Lak87]). We can therefore call $\Omega^{r,n}$ the scheme of *"n-complete homomorphisms of rank r"*.

If $p \geq 1$ is an integer, any map $\iota : \{0,\ldots,p\} \to \{0,\ldots,n\}$ induces naturally an affine map of

$$S^{r,p} = \{(i_0,\ldots,i_n) \in \mathbb{N}^{n+1} \mid i_0 + \cdots + i_n = r\}$$

by

$$(j_\beta)_{0\le\beta\le p} \mapsto \big(i_\alpha = \sum_{\iota(\beta)=\alpha} j_\beta\big)_{0\le\alpha\le n};$$

it is an identification to a face when ι is injective and a projection when ι is surjective. For any entire tiling $\underline{S}$ of $S^{r,n}$, the functions $S^{r,n}\to\mathbb{R}$ which are in the convex cone $\mathcal{C}^{r,n}_{\underline{S}}$ induce on $S^{r,p}$ functions which are in the convex cone $\mathcal{C}^{r,p}_{\underline{S}'}$ associated to entire tiling $\underline{S}'$ of $S^{r,p}$ induced by $\underline{S}$. Thus ι induces an equivariant morphism

$$\iota^* : \mathcal{A}^{r,n} \to \mathcal{A}^{r,p}.$$

On the other hand, for each

$$(j_\beta)_{0\le\beta\le p} \in S^{r,p}$$

and

$$(i_\alpha = \sum_{\iota(\beta)=\alpha} j_\beta)_{0\le\alpha\le n} \in S^{r,n}$$

its image, we have a surjective homomorphism

$$\bigotimes_{0\le\beta\le p} \Lambda^{j_\beta}\mathbb{A}^r \to \bigotimes_{0\le\alpha\le n} \Lambda^{i_\alpha}\mathbb{A}^r$$

and by duality an injective homomorphism

$$\bigotimes_{0\le\alpha\le n} \Lambda^{i_\alpha}\mathbb{A}^r \to \bigotimes_{0\le\beta\le p} \Lambda^{j_\beta}\mathbb{A}^r$$

so that ι induces a morphism

$$\begin{aligned}\iota^* : \mathbb{G}_m\backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} &\Big[(\Lambda^{i_0}\mathbb{A}^r\otimes\cdots\otimes\Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]\\ &\to \mathbb{G}_m\backslash \prod_{(i_0,\dots,i_p)\in S^{r,p}} \Big[(\Lambda^{i_0}\mathbb{A}^r\otimes\cdots\otimes\Lambda^{i_p}\mathbb{A}^r) - \{0\}\Big].\end{aligned}$$

And we verify immediately that the product morphism sends $\Omega^{r,n}_\emptyset$ to $\Omega^{r,p}_\emptyset$ via the obvious morphism

$$\iota^* : (\mathrm{GL}_r^{n+1}\times\mathbb{G}_m^{S^{r,n}})/(\mathrm{GL}_r\times\mathbb{G}_m^{n+1}) \to (\mathrm{GL}_r^{p+1}\times\mathbb{G}_m^{S^{r,p}})/(\mathrm{GL}_r\times\mathbb{G}_m^{p+1})$$

and hence induces a morphism $\iota^* : \Omega^{r,n}\to\Omega^{r,p}$ which is included in the commutative diagram:

$$\begin{array}{ccccccc}(\mathrm{GL}_r^{n+1}\times\mathbb{G}_m^{S^{r,n}})/(\mathrm{GL}_r\times\mathbb{G}_m^{n+1}) & \hookrightarrow & \Omega^{r,n} & \longrightarrow & \mathcal{A}^{r,n} & \hookleftarrow & \mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m^{n+1}\\ \downarrow & & \downarrow & & \downarrow & & \downarrow\\ (\mathrm{GL}_r^{p+1}\times\mathbb{G}_m^{S^{r,p}})/(\mathrm{GL}_r\times\mathbb{G}_m^{p+1}) & \hookrightarrow & \Omega^{r,p} & \longrightarrow & \mathcal{A}^{r,p} & \hookleftarrow & \mathbb{G}_m^{S^{r,p}}/\mathbb{G}_m^{p+1}\end{array}$$

And ass $\iota^* : \Omega^{r,n}\to\Omega^{r,p}$ is equivariant, it induces finally a morphism $\iota^* : \overline{\Omega}^{r,n}\to\overline{\Omega}^{r,p}$.

The family of $\Omega^{r,n}$ [resp. of $\overline{\Omega}^{r,n}$], $n\ge 1$, with all these induced morphisms is a simplicial scheme that extends the one of $(\mathrm{GL}_r^{n+1}\times\mathbb{G}_m^{S^{r,n}})/(\mathrm{GL}_r\times\mathbb{G}_m^{n+1})$ [resp. of $\mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r$ which is none other than the classifier of PGL_r].

For $n\ge 2$ and $p=1$, let us describe to *iota* the set of $n+1$ injections $\{0,1\}\to\{0,\dots,n\}$ given by

$$0\mapsto 0,\ 1,\dots,\ n-1,\ n$$

and

$$1 \mapsto 1,\ 2, \ldots, n,\ 0.$$

Then the induced morphism

$$\overline{\Omega}^{r,n} \to (\overline{\Omega}^{r,1})^{n+1}$$

sends $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$ to $(\mathrm{PGL}_r^2 / \mathrm{PGL}_r)^{n+1} \cong \mathrm{PGL}_r^{n+1}$ in such a way that identifies with

$$\{(g_0, \ldots, g_n) \mid g_0 g_1 \cdots g_n = 1\}.$$

Thus, $\overline{\Omega}^{r,n}$ realizes a compactification of morphism of multiplication of n elements of PGL_r.

1.3 Smoothness and modular descriptions

We will prove:

Theorem 1.6. (i) *The morphism $\Omega^{r,n} \to \mathcal{A}^{r,n}$, which is equivariant under the action of torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$, is surjective and smooth of relative dimension nr^2.*

(ii) *If $\iota : \{0, \ldots, p\} \to \{0, \ldots, n\}$ is an injective map, the induced morphism $\Omega^{r,n} \to \Omega^{r,p} \times_{\mathcal{A}^{r,p}} \mathcal{A}^{r,n}$, which is equivariant under the action of torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$, is smooth of relative dimension $(n-p)r^2$.*

As the morphism $\Omega^{r,n} \to \mathcal{A}^{r,n}$ is equivariant, we see that, for any convex entire tiling $\underline{S}$ of $S^{r,n}$, the preimage $\Omega_{\underline{S}}^{r,n}$ in $\Omega^{r,n}$ of the orbit $\mathcal{A}_{\underline{S}}$ in $\mathcal{A}^{r,n}$ is canonically isomorphic to $\mathbb{G}_m^{\underline{S}} \backslash (\mathbb{G}_m^{S^{r,n}} \times \mathbb{G}_{\underline{S}}^{r,n})$ where $\mathbb{G}_{\underline{S}}^{r,n}$ denotes the fiber of $\Omega^{r,n}$ above marked point $\alpha_{\underline{S}}$ of $\mathcal{A}_{\underline{S}}$ and $\mathbb{G}_m^{\underline{S}}$ the sub-torus of $\mathbb{G}_m^{\underline{S}}$ stabilizer of this point.

Of course, $\Omega_{\underline{S}}^{r,n}$ constitutes a stratification of $\Omega^{r,n}$ as disjoint union of smooth locally closed subschemes. And the closure of each stratum $\Omega_{\underline{S}}^{r,n}$ is the union of $\Omega_{\underline{S}'}^{r,n}$ on the set of convex entire tilings $\underline{S}'$ refining $\underline{S}$.

Likewise, the quotients $\overline{\Omega}_{\underline{S}}^{r,n}$ of $\Omega_{\underline{S}}^{r,n}$ by the free action of torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$ constitutes a stratification of $\overline{\Omega}^{r,n}$ as disjoint union of smooth locally closed subschemes.

Finally, we want to describe in modular terms the fibers $\mathrm{Gr}_{\underline{S}}^{r,n}$ of $\Omega^{r,n}$ above marked points $\alpha_{\underline{S}}$ of $\mathcal{A}^{r,n}$. For this purpose, let's denote $E_0, \ldots, E_n$ the $n+1$ factors $\mathbb{A}^r$ of $(\mathbb{A}^r)^{n+1} = E$ and, for any subset I of $\{0, \ldots, n\}$, $E_I = \oplus_{\alpha \in I} E_\alpha$.

We will start with the following lemma:

Lemma 1.7. *Let F be a subspace of dimension r of $E = (\mathbb{A}^r)^{n+1}$ defined on a field and represented by a nonzero tuple $(u_{i_0,\ldots,i_n})$ in*

$$\Lambda^r (\mathbb{A}^r)^{n+1} = \bigoplus_{(i_0,\ldots,i_n) \in S^{r,n}} \Lambda^{i_0} \mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n} \mathbb{A}^r.$$

Then the family of integers $d_I = \dim(F \cap E_I)$ is convex in the sense of Definition 1.1 and the subset of $S^{r,n}$

$$\{(i_0, \ldots, i_n) \mid u_{i_0,\ldots,i_n} \neq 0\}$$

is none other than the entire convex polyhedron

$$\{(i_0, \ldots, i_n) \mid \sum_{\alpha \in I} i_\alpha \geq d_I,\ \forall\ I\}.$$

If then S is a entire convex polyhedron of $S^{r,n}$ defined by a convex family of integers $d_I \geq 0$, we are led to introduce the scheme $\mathrm{Gr}_S^{r,n}$ of tuples $(u_{i_0,\ldots,i_n})_{(i_0,\ldots,i_n)\in S^{r,n}}$ in

$$\mathbb{G}_m \backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

which, completed by 0 at indexes of $S^{r,n} - S$, is in the Grassmannian $\mathrm{Gr}^{r,n}$. This is a locally closed subscheme of $\mathrm{Gr}^{r,n}$ and according to Lemma 1.7, it classifies the subspaces F of dimension r in $E = (\mathbb{A}^r)^{n+1}$ such that $\dim(F \cap E_I) = d_I$, $\forall\ I$. In addition, when S runs through the finite set of entire convex polyhedron of $S^{r,n}$, $\mathrm{Gr}_S^{r,n}$ constitutes a stratification of $\mathrm{Gr}^{r,n}$. These strata of the Grassmannian are variants of those studied in the mathematical literature under the name of "thin Schubert cells" (see for example [Gel+87]).

Let us also state:

Lemma 1.8. *For S be a entire convex polyhedron of $S^{r,n}$ and S' a face of S, the restriction $(u_s)_{s\in S} \mapsto (u_s)_{s\in S'}$ induces a morphism of $\mathrm{Gr}_S^{r,n}$ to $\mathrm{Gr}_{S'}^{r,n}$. When S' is defined in S by an equation of the form*

$$\sum_{\alpha\in I} i_\alpha = d_I \quad \textit{with} \quad d_I = \min_{(i_0,\ldots,i_n)\in S} \Big\{\sum_{\alpha\in I} i_\alpha\Big\},$$

and in particular when S' is a boundary of S, this morphism associates with subspaces F of E which are in $\mathrm{Gr}_S^{r,n}$ the subspaces

$$(F \cap E_I) \oplus F/(F \cap E_I) \subseteq E_I \oplus E/E_I = E.$$

We will give fibers $\mathrm{Gr}_{\underline{S}}^{r,n}$ the following modular description:

Theorem 1.9. *For any convex entire tiling $\underline{S}$ of $S^{r,n}$, the fiber $\mathrm{Gr}_{\underline{S}}^{r,n}$, which is smooth of dimension nr^2, is the closed subscheme of*

$$\mathbb{G}_m \backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

of tuples $(u_{i_0,\ldots,i_n})_{(i_0,\ldots,i_n)\in S^{r,n}}$ such that, for any tile S of $\underline{S}$, the restraint tuple $(u_{i_0,\ldots,i_n})_{(i_0,\ldots,i_n)\in S}$ are in $\mathrm{Gr}_S^{r,n}$.

It classifies the families (F_S) of subspaces of dimension r in E indexed by tiles S of $\underline{S}$ such that:

- *for any tile S of $\underline{S}$ and any subset I of $\{0,\ldots,n\}$,*

$$\dim(F_S \cap E_I) = \min_{(i_0,\ldots,i_n)\in S} \Big\{\sum_{\alpha\in I} i_\alpha\Big\},$$

- *for any tiles S, S' of $\underline{S}$ having in common a boundary with equation*

$$\sum_{\alpha\in I} i_\alpha = d_I$$

where

$$d_I = \min_{(i_0,\ldots,i_n)\in S} \Big\{\sum_{\alpha\in I} i_\alpha\Big\} = \max_{(i_0,\ldots,i_n)\in S'} \Big\{\sum_{\alpha\in I} i_\alpha\Big\}$$

and if $J = \{0,\ldots,n\} - I$,

$$F_S \cap E_I = F_{S'}/(F_{S'} \cap E_J), \quad F_S/(F_S \cap E_I) = F_{S'} \cap E_J.$$

Chapter 2

entire convex polyhedrons and convex entire tilings

We will give here the demonstrations of the results that are used for the construction of Section 1.1.

2.1 An equivalent definition of entire convex polyhedrons

Let's start with the following lemma:

Lemma 2.1. *Let (d_I) be a family of integers indexed by subsets of $\{0,\ldots,n\}$ and convex, with hence*

$$d_\emptyset = 0,\ d_{\{0,\ldots,n\}} = r,\ d_I + d_J \le d_{I\cap J} + d_{I\cup J}, \forall I, J.$$

And let

$$S = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{0\le\alpha\le n} x_\alpha = r \text{ and } \sum_{\alpha\in I} x_\alpha \ge d_I,\ \forall\ I\}$$

be the associated entire convex polyhedron.

Then, if $(x_0,\ldots,x_n)$ is a point of S and I, J two subsets of $\{0,\ldots,n\}$ such that

$$\sum_{\alpha\in I} x_\alpha = d_I \text{ and } \sum_{\alpha\in J} x_\alpha = d_J,$$

we have

$$d_I + d_J = d_{I\cap J} + d_{I\cup J}$$

and

$$\sum_{\alpha\in I\cap J} x_\alpha = d_{I\cap J},\ \sum_{\alpha\in I\cup J} x_\alpha = d_{I\cup J}.$$

Proof. To see this, it is enough to combine the two equalities of the hypothesis with the inequalities

$$\sum_{\alpha\in I\cup J} x_\alpha \ge d_{I\cup J},\quad \sum_{\alpha\in I\cap J} x_\alpha \ge d_{I\cap J},$$
$$d_I + d_J \le d_{I\cap J} + d_{I\cup J}$$

and with the equality

$$\sum_{\alpha\in I} x_\alpha + \sum_{\alpha\in J} x_\alpha = \sum_{\alpha\in I\cap J} x_\alpha + \sum_{\alpha\in I\cup J} x_\alpha.$$

□

This lemma allows us to prove the necessity of the condition given by the following proposition:

Proposition 2.2. *A convex polyhedron S of the space is entire if and only if for any sequence*

$$S = S_{l_0},\ S_{l_0+1},\ \ldots,\ S_n$$

constituted of convex polyhedra S_l, $l_0 \le l \le n$, of codimension l, each of which is a boundary of the previous one, there exists a permutation τ of $\{0,\ldots,n\}$, a *permutation σ of $\{0,\dots,n\}$ and integers $d_1,\dots,d_n \in \mathbb{Z}$ such that, for any l, $l_0 \leq l \leq n$, barycentric coordinates $x_0,\dots,x_n$ of points of S_l satisfying the equations:*

$$\begin{cases} x_{\tau(\sigma(1))} + x_{\tau(\sigma(1)+1)} + \cdots + x_{\tau(n)} = d_1, \\ x_{\tau(\sigma(2))} + x_{\tau(\sigma(2)+1)} + \cdots + x_{\tau(n)} = d_2, \\ \vdots \\ x_{\tau(\sigma(l))} + x_{\tau(\sigma(l)+1)} + \cdots + x_{\tau(n)} = d_l. \end{cases}$$

Proof. Let's start by proving the necessity of this condition.

Given that

$$S = S_{l_0},\ S_{l_0+1},\ \dots,\ S_n$$

be a sequence as in the statement, we can find a sequence

$$I_1,\dots,I_n$$

of subsets of $\{0,\dots,n\}$ such that the affine subspace generated by each S_l, $l_0 \leq l \leq n$, is defined by the system of equations

$$\sum_{\alpha\in I_m} x_\alpha = d_{I_m}, \quad 1 \leq m \leq l.$$

And according to Lemma 2.1, we can suppose that for any m, m' we have

$$I_m \subseteq I_{m'} \text{ or } I_m \supseteq I_{m'} \text{ or } I_m \cap I_{m'} = \emptyset \text{ or } I_m \cup I_{m'} = \{0,\dots,n\}.$$

Even if you replace I_m by $\{0,\dots,n\} - I_m$ for certain m, $1 \leq m \leq n$, we see that the subspaces generated by S_l, $l_0 \leq l \leq n$, are defined by systems of equations of the form

$$\sum_{\alpha\in J_m} x_\alpha = d_m, \quad 1 \leq m \leq l,$$

where $J_1,\dots,J_n$ is a sequence of subsets of $\{0,\dots,n\}$ such that for any m, m' we have $J_m \subseteq J_{m'}$ or $J_{m'} \subseteq J_m$ and d_m are integers. This is the form requested in the condition of the statement.

As for the converse, we will show it at the same time with the following lemma:

Lemma 2.3. *Let S be a convex polyhedron of the form*

$$S = \{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{0\leq\alpha\leq n} x_\alpha = r \text{ and } \sum_{\alpha\in I} x_\alpha \geq d_I,\ \forall\ I\}$$

where, for any subset I,

$$d_I = \min_{(x_0,\dots,x_n)\in S} \left\{ \sum_{\alpha\in I} x_\alpha \right\}.$$

Then S is entire if and only if d_I are elements of $\mathbb{Z}$ and for any subsets

$$I_1 \supseteq I_2 \supseteq I_2 \supseteq \cdots \supseteq I_l,$$

three exists a point $(x_0,\dots,x_n)$ of S which realizes simultaneously the minimums of

$$\sum_{\alpha\in I_1} x_\alpha,\ \sum_{\alpha\in I_2} x_\alpha,\ \dots,\ \sum_{\alpha\in I_l} x_\alpha.$$

Proof of Lemma 2.3 and finally of Proposition 2.2. Let us first show that if the condition of Lemma 2.3 is satisfied, S is entire. It suffices to show that for any subsets I, J of $\{0,\ldots,n\}$, we have

$$d_I + d_J \leq d_{I\cap J} + d_{I\cup J}.$$

Now there is by hypothesis a point $(x_0,\ldots,x_n)$ of S such that

$$\sum_{\alpha\in I\cap J} x_\alpha = d_{I\cap J},\ \sum_{\alpha\in I} x_\alpha = d_I,\ \sum_{\alpha\in I\cup J} x_\alpha = d_{I\cup J}.$$

We deduce as desired

$$d_J \leq \sum_{\alpha\in J} x_\alpha = d_{I\cup J} + d_{I\cap J} - d_I.$$

It is obvious on the other hand that if a convex polyhedron S satisfies the condition of Proposition 2.2, it has the form of Lemma 2.3 and

$$d_I = \min_{(x_0,\ldots,x_n)\in S} \left\{ \sum_{\alpha\in I} x_\alpha \right\}$$

are integers. It only remains for us to prove that then also and for all subsets

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_l,$$

minimums of

$$\sum_{\alpha\in I_1} x_\alpha,\ \sum_{\alpha\in I_2} x_\alpha,\ \ldots,\ \sum_{\alpha\in I_l} x_\alpha$$

are realized simultaneously on S.

Let us proceed by recurrence on n, assuming the result already satisfied when the space is of dimension $< n$.

Let us first consider the case where S is of dimension $< n$. As by hypothesis S satisfies the condition of Proposition 2.2, there exists then a non trivial partition

$$K_1 \amalg \cdots \amalg K_k$$

of $\{0,\ldots,n\}$ and integers $r_1,\ldots,r_k$ of sum $r_1+\cdots+r_k = r$ such that the subspace generated by S is defined by equations

$$\sum_{\alpha\in K_1} x_\alpha = r_1,\ \sum_{\alpha\in K_2} x_\alpha = r_2,\ \ldots,\ \sum_{\alpha\in K_k} x_\alpha = r_k,$$

and S is of the form $S_1\times\cdots\times S_k$ where $S_1,\ldots,S_k$ are entire tiles in the spaces

$$\{(x_\alpha)_{\alpha\in K_1} \mid \sum x_\alpha = r_1\},\ldots,\{(x_\alpha)_{\alpha\in K_k} \mid \sum x_\alpha = r_k\}.$$

In consequence, we can write, for $1\leq l'\leq l$,

$$\min_{(x_\alpha)\in S}\left\{\sum_{\alpha\in I_{l'}} x_\alpha\right\} = \min_{(x_\alpha)\in S_1}\left\{\sum_{\alpha\in I_{l'}\cap K_1} x_\alpha\right\} + \cdots + \min_{(x_\alpha)\in S_k}\left\{\sum_{\alpha\in I_{l'}\cap K_k} x_\alpha\right\}.$$

The result for S can be deduced from the one already known for $S_1,\ldots,S_k$ according to the hypothesis of recurrence.

It remains to treat the case where S is of dimension n i.e. is a tile.

let's first assume that the equation

$$\sum_{\alpha\in I_1} x_\alpha = d_{I_1}$$

(where $d_{I_1} = \min_{(x_\alpha)\in S}\left\{\sum_{\alpha\in I_1} x_\alpha\right\}$) defines a boundary of S. This one also satisfies the condition of Proposition 2.2 and it is of the form $S'\times S''$ where S' and S'' are two entire tiles in the spaces

$$\{(x_\alpha)_{\alpha\in I_1} \mid \sum x_\alpha = d_{I_1}\}$$

and

$$\{(x_\alpha)_{\alpha\notin I_1} \mid \sum x_\alpha = r - d_{I_1}\};$$

more precisely, it is defined in this hyperplane by a certain number of inequalities of the form

$$\sum_{\alpha\in K} x_\alpha \geq d_K$$

where subsets K must satisfy

$$K\subseteq I_1 \text{ or } K\supseteq I_1 \text{ or } K\cap I_1=\emptyset \text{ or } K\cup I_1=\{0,\dots,n\}.$$

The tile S' is defined by those inequalities that correspond to subsets K such that $K\subseteq I_1$ or $K\cup I_1=\{0,\dots,n\}$. They are of the form

$$\sum_{\alpha\in K} x_\alpha \geq d_K \text{ or } \sum_{\alpha\in I_1\cap K} x_\alpha \geq d_{I_1}+d_K-r$$

and in both cases they are already satisfied on the tile S in a whole. We deduce that the minima on S of $\sum_{\alpha\in I_{l'}} x_\alpha$, $2\leq l'\leq l$, are realized on $S'\times S''$ simultaneously according to the result proved for $S'\times S''$.

Let's consider the last case where the equation

$$\sum_{\alpha\in I_1} x_\alpha = d_{I_1}$$

defines in S a face of codimension $k\geq 2$. This face is of the form

$$S_0\otimes S_1\otimes\cdots\otimes S_k$$

where $S_0,\dots,S_k$ are tiles in the spaces

$$\{(x_\alpha)_{\alpha\in K_0} \mid \sum x_\alpha = r_0\},\dots,\{(x_\alpha)_{\alpha\in K_k} \mid \sum x_\alpha = r_k\}$$

with

$$K_0 \amalg K_1 \amalg\cdots\amalg K_k$$

a partition of $\{0,\dots,n\}$ and $r_0,\dots,r_k$ integers of sum $r_0+r_1+\cdots+r_k=r$. In addition, I_1 is compatible with this partition in the sense that if we renumber we have

$$I_1 = K_0\amalg K_1\amalg\cdots\amalg K_{k'}$$

for certain $k'<k$.

The face

$$S_0\otimes S_1\otimes\cdots\otimes S_k$$

is the intersection of boundaries of S that contains; this is defined by equations of the form

$$\sum_{\alpha\in K} x_\alpha = d_K$$

where K is a union of certain K_m and

$$d_K = \sum_{K_m\subseteq K} r_m.$$

And for any $m \leq k'$, we can choose among K which appear at least one K'_m such that $K_m \subseteq K'_m$; denote B_m the boundary of S that corresponds to it.

For $2 \leq l' \leq l$, we have certainly

$$\min_{(x_\alpha)\in S_0\times\cdots\times S_k}\left\{\sum_{\alpha\in I_{l'}} x_\alpha\right\} \geq \min_{(x_\alpha)\in S}\left\{\sum_{\alpha\in I_{l'}} x_\alpha\right\} \geq \sum_{0\leq m\leq k'} \min_{(x_\alpha)\in S}\left\{\sum_{\alpha\in I_{l'}\cap K_m} x_\alpha\right\}.$$

However, according to the already known result for boundaries of S and the hypothesis of recurrence, we have for any $m \leq k'$,

$$\min_{(x_\alpha)\in S}\left\{\sum_{\alpha\in I_{l'}\cap K_m} x_\alpha\right\} = \min_{(x_\alpha)\in B_m}\left\{\sum_{\alpha\in I_{l'}\cap K_m} x_\alpha\right\} = \min_{(x_\alpha)\in S_0\times\cdots\times S_k}\left\{\sum_{\alpha\in I_{l'}\cap K_m} x_\alpha\right\}.$$

As $I_{l'} \subseteq I_1$, we have

$$\min_{(x_\alpha)\in S_0\times\cdots\times S_k}\left\{\sum_{\alpha\in I_{l'}} x_\alpha\right\} = \sum_{0\leq m\leq k'} \min_{(x_\alpha)\in S_0\times\cdots\times S_k}\left\{\sum_{\alpha\in I_{l'}\cap K_m} x_\alpha\right\}.$$

We conclude according to the result already known for $S_0 \times \cdots \times S_k$.

This finishes the proofs. □

□

2.2 The properties of entire convex polyhedrons

We give here the proof of Lemma 1.2 of Section 1.1.

(i) Let then (d_I) be a family of integers indexed by subsets I of $\{0,\ldots,n\}$ and convex in the sense that

$$d_\emptyset = 0,\ d_{\{0,\ldots,n\}} = r,\ d_I + d_J \leq d_{I\cap J} + d_{I\cup J}, \forall I, J.$$

We want to show by recurrence on the dimension n of the space that the associated convex polyhedron

$$S = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{0\leq\alpha\leq n} x_\alpha = r \text{ and } \sum_{\alpha\in I} x_\alpha \geq d_I,\ \forall\ I\}$$

satisfying

$$d_I = \min_{(x_0,\ldots,x_n)\in S}\left\{\sum_{\alpha\in I} x_\alpha\right\},\ \forall\ I,$$

and in particular is not empty. This is obvious when $n = 1$. Suppose then that $n \geq 2$ and that this result is already verified of dimension $< n$.

If I is a non trivial subset of $\{0,\ldots,n\}$ and J its complementary, it results from Lemma 2.1 of Section 2.1 that the intersection of S with the hyperplane

$$\{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{\alpha\in I} x_\alpha = d_I\}$$

is of the form $S' \times S''$ where S' and S'' are two convex polyhedra in the spaces

$$\{(x_\alpha)_{\alpha\in I} \mid \sum x_\alpha = d_I\} \text{ and } \{(x_\alpha)_{\alpha\in J} \mid \sum x_\alpha = r - d_I\}$$

defined respectively by the inequalities

$$\sum_{\alpha\in K} \geq d_K,\ \forall\ K \subseteq I$$

and

$$\sum_{\alpha\in K} x_\alpha \geq d_{I\cup K} - d_I, \ \forall\ K \subseteq J.$$

Now the two families of integers $(d_K)_{K\subseteq I}$ and $(d_{I\cup K} - d_I)_{K\subseteq J}$ are convex. According to the hypothesis of recurrence S' and S'' are nonempty and thus

$$d_I = \min_{(x_0,\ldots,x_n)\in S} \left\{ \sum_{\alpha\in I} x_\alpha \right\}$$

as we wanted.

(ii) That the faces of an entire convex polyhedron are still entire convex polyhedra follows immediately from their characterization given by Proposition 2.2 of Section 2.1.

(iii) The assertion (ii) says in particular that the vertices of an entire convex polytope are integer points. As any convex polyhedron is generated by its vertices, we see that an entire convex polytope is generated by its integer points.

(iv) We want to prove that in any entire tile S of the space of dimension n, we can choose $n+1$ points which generated the the complete lattice of the integer points.

Let's choose any sequence

$$S = S_0,\ S_1,\ \ldots,\ S_n$$

of faces S_l, $0 \leq l \leq n$, of S of codimension l each of which is a boundary of the previous one. According to Proposition 2.2 of Section 2.1, there exists a permutation τ of $\{0,\ldots,n\}$ and a permutation σ of $\{0,\ldots,n\}$ such that, for any l, $1 \leq l \leq n$, and denoting s_n the unique point of S_n, points s of S_l satisfies the equations with the barycentric coordinates:

$$\begin{cases} (x_{\tau(\sigma(1))} + x_{\tau(\sigma(1)+1)} + \cdots + x_{\tau(n)})(s - s_n) = 0, \\ (x_{\tau(\sigma(2))} + x_{\tau(\sigma(2)+1)} + \cdots + x_{\tau(n)})(s - s_n) = 0, \\ \vdots \\ (x_{\tau(\sigma(l))} + x_{\tau(\sigma(l)+1)} + \cdots + x_{\tau(n)})(s - s_n) = 0. \end{cases}$$

For any l, $1 \leq l \leq n$, we can choose in S_{l-1} an integer point s_{l-1} such that

$$(x_{\tau(\sigma(l))} + x_{\tau(\sigma(l)+1)} + \cdots + x_{\tau(n)})(s_{l-1} - s_n) = \pm 1.$$

Indeed, there is certainly in S_{l-1} a sequence of faces

$$S_{l-1} = S'_{l-1},\ S'_l,\ \ldots,\ S'_{n-1},\ S'_n = S_n$$

each of which is a boundary of the previous one and such that each $S'_{l'} \cap S_l$, $l \leq l' \leq n$, is of codimension $l'+1$. According to Proposition 2.2 of Section 2.1, S'_{n-1} has a (unique) point s_{l-1} satisfying the above equation.

The family $s_0,\ldots,s_n$ answers the question asked. In fact, the matrix

$$(x_{\tau(\sigma(l))} + x_{\tau(\sigma(l)+1)} + \cdots + x_{\tau(n)})(s_{l'-1} - s_n),\ 1 \leq l, l' \leq n,$$

has its integer coefficients, zero above the diagonal and equal to $\pm$ on the diagonal.

(v) Let S be a tile of the space of dimension n which admits a tiling constituted of a finite set $\{S'\}$ of tiles. We remark that if

$$S = S_0,\ S_1,\ \ldots,\ S_n$$

is a sequence of faces of S each of which is a boundary of the previous one, then among the tiles S' of tiling, there is one for which the sequence

$$S' = S' \cap S_0, \; S' \cap S_1, \; \dots, \; S' \cap S_n$$

is constituted of faces of S' each of which is a boundary of the previous one.

It is thus resulted from Proposition 2.2 of Section 2.1 that if the elements of tiling $\{S'\}$ of S are entire, S is also entire.

This finishes the proof of Lemma 1.2 of Section 1.1.

2.3 The properties of cones of convex functions on the simplex

We give here the demonstration of Proposition 1.3 of Section 1.1.

(i) According to Lemma 1.2(ii) of Section 1.1, the faces of any entire tile of $S^{r,n}$ are entire convex polyhedra. From this it follows immediately that all convex cones $\mathcal{C}_{\underline{S}}^{P} r, n$ are contained in the cone $\mathcal{C}^{r,n}$.

Conversely, consider $v : S^{r,n} \to \mathbb{R}$ a function which is in the cone $\mathcal{C}^{r,n}$.

For any subset S of $S^{r,n}$, denote $\overline{S}$ the convex polyhedron of

$$\mathbb{R}^{r,n} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \dots + x_n = r\}$$

that it generates, and for any affine map $l : S^{r,n} \to \mathbb{R}$, denote $\bar{l} : \overline{S^{r,n}} \to \mathbb{R}$ the unique affine map which extends it.

Let's have l describe the set $\{l\}$ of affine maps $S^{r,n} \to \mathbb{R}$ satisfying $l \le v$. Then $\overline{v} = \sup_{l \in \{l\}} \{\bar{l}\}$ is a convex application $\overline{S^{r,n}} \to \mathbb{R}$ which is written as the upper bound of a finite set of affine maps and which satisfies $\overline{v}(s) \le v(s)$, $\forall\ s \in S^{r,n}$. There exists a unique tiling $\underline{X}$ of $\overline{S^{r,n}}$ such that at any tile X of $\underline{X}$ is attached an affine map $l_X : \overline{S^{r,n}} \to \mathbb{R}$ satisfying $l_X \le \overline{v}$ and

$$X = \{x \in \overline{S^{r,n}} \mid l_X(x) = \overline{v}(x)\}.$$

In addition, any such tile X is generated as convex polyhedron by its finite subset

$$S = \{s \in S^{r,n} \mid l_X(s) = v(s)\}.$$

But since the function v is by hypothesis in the cone $\mathcal{C}^{r,n}$, we see that S is an entire tile of $S^{r,n}$ and $X = \overline{S}$.

Then the set of these entire tiles S constitutes the unique entire tiling $\underline{S}$ of $S^{r,n}$ such that the convex cone $\mathcal{C}_{\underline{S}}^{r,n}$ contains v.

(ii) For any convex entire tiling $\underline{S}$ of $S^{r,n}$, the closure $\overline{\mathcal{C}_{\underline{S}}^{r,n}}$ of convex cone $\mathcal{C}_{\underline{S}}^{r,n}$ is the set of functions $v : S^{r,n} \to \mathbb{R}$ such that for any tile S of $\underline{S}$ there exists an affine map $l_S : S^{r,n} \to \mathbb{R}$ satisfying $l_S \le v$ and

$$S \subseteq \{s \in S^{r,n} \mid l_S(s) = v(s)\}.$$

For v a such function and S a tile of $\underline{S}$, the set

$$\{s \in S^{r,n} \mid l_S(s) = v(s)\}$$

is the trace of a tile of $\overline{S^{r,n}}$ which is a union of tiles of $\underline{S}$; according to Lemma 1.2(v) of Section 1.1, this is an entire tile. This proves that $\overline{\mathcal{C}_{\underline{S}}^{r,n}}$ is the (disjoint) union of $\overline{\mathcal{C}_{\underline{S}'}^{r,n}}$ with $\underline{S}'$ describing the set of entire tilings coarser than $\underline{S}$.

In order to show that $\overline{\mathcal{C}_{\underline{S}}^{r,n}}$ is a rational polyhedral convex cone, it suffice to see that it is defined by a finite number of equations and of linear inequalities with coefficients in $\mathbb{Z}$. For this, let's choose in each tile S of $\underline{S}$ a generating family

$s_0, \ldots, s_n$ (it exists according to Lemma 1.2(iv) of Section 1.1). Any point s of $S^{r,n}$ is written in a unique way as the barycenter of $s_0, \ldots, s_n$ with integer coefficients $j_0, \ldots, j_n$ of total weight $j_0 + j_1 + \cdots + j_n = 1$. Then the cone $\overline{\mathcal{C}^{r,n}_{\underline{S}}}$ is defined in the space of functions $v : S^{r,n} \to \mathbb{R}$ by the finite set of equations and inequalities of the form

$$v(s) = j_0 v(s_0) + j_1 v(s_1) + \cdots + j_n v(s_n)$$

if s is in the same tile S as $s_0, \ldots, s_n$, and

$$v(s) \geq j_0 v(s_0) + j_1 v(s_1) + \cdots + j_n v(s_n)$$

in the general case.

As $\overline{\mathcal{C}^{r,n}_{\underline{S}}}$ is the disjoint union of $\mathcal{C}^{r,n}_{\underline{S}'}$ with ulS' more coarse than $\underline{S}$, it is obvious now that the faces of $\overline{\mathcal{C}^{r,n}_{\underline{S}}}$ are $\overline{\mathcal{C}^{r,n}_{\underline{S}'}}$.

(iii) Consider than $\underline{S}$ and $\underline{S}'$ two convex entire tilings of $S^{r,n}$ and $\underline{S}_1, \ldots, \underline{S}_k$ the finite family of convex entire tilings more coarse than both $\underline{S}$ and $\underline{S}'$. The intersection $\underline{S} \vee \underline{S}'$ of these is a tiling of $\overline{S^{r,n}}$ whose tiles are union of tiles of $\underline{S}$ or also of $\underline{S}'$. Again according to Lemma 1.2(v) of Section 1.1, these tiles are integers and $\underline{S} \vee \underline{S}'$ is an entire tiling of $S^{r,n}$. Finally, if we choose functions $v_1, \ldots, v_k$ in the convex cones $\mathcal{C}^{r,n}_{\underline{S}_1}$, ..., $\mathcal{C}^{r,n}_{\underline{S}_k}$, the function $v_1 + \cdots + v_k$ is in the convex cone $\mathcal{C}^{r,n}_{\underline{S} \vee \underline{S}'}$ which thus is not empty and $\underline{S} \vee \underline{S}'$ is a convex entire tiling as we wanted.

This finishes the proof of Proposition 1.3 of Section 1.1.

Chapter 3

Stratification of the Grassmannian and schemes of glued graphs

3.1 The entire convex polyhedron attached to a graph. The restrictions to boundaries

We demonstrate here the results needed to define and interpret in modular terms the strata $\mathrm{Gr}_S^{r,n}$ of the Grassmannian $\mathrm{Gr}^{r,n}$ associated to entire convex polyhedra S of $S^{r,n}$ as well as morphisms $\mathrm{Gr}_S^{r,n} \to \mathrm{Gr}_{S'}^{r,n}$ of restriction to boundaries.

Proof of Lemma 1.7 of Section 1.3. Let then F be a subspace of dimension r of $(\mathbb{A}^r)^{n+1} = E = E_0 \oplus \cdots \oplus E_n$. For any subset I of $\{0, \dots, n\}$, we denote d_I the dimension of the intersection of F with

$$E_I = \bigoplus_{\alpha \in I} E_\alpha.$$

Of course, we have $d_\emptyset = 0$, $d_{\{0,\dots,n\}} = r$ and for all subsets I, J, the subspaces $F \cap E_I$ and $F \cap E_J$ are contained in $F \cap E_{I \cup J}$ and have for intersection $F \cap E_{I \cap J}$ so that

$$d_I + d_J - d_{I \cap J} \leq d_{I \cup J}.$$

Thus, the family (d_I) is convex.

If $u = (u_{i_0,\dots,i_n})$ is a direction vector of the line that represents F in the space

$$\Lambda^r E = \bigoplus_{(i_0,\dots,i_n) \in S^{r,n}} \Lambda^{i_0} E_0 \otimes \cdots \otimes \Lambda^{i_n} E_n,$$

we need to show that a tuple $(i_0, \dots, i_n) \in S^{r,n}$ satisfying $u_{i_0,\dots,i_n} \neq 0$ if and only if

$$\sum_{\alpha \in I} i_\alpha \geq d_I, \quad \forall\ I.$$

Now, for any subset I of $\{0, \dots, n\}$, we have a canonical isomorphism

$$\Lambda^r F \cong \Lambda^{d_I}(F \cap E_I) \otimes \Lambda^{r-d_I}(F/F \cap E_I),$$

from which it results

$$d_I = \min\left\{ \sum_{\alpha \in I} i_\alpha \mid u_{i_0,\dots,i_n} \neq 0 \right\}.$$

It suffices thus to prove that if $(j_0, \dots, j_n)$ is a tuple of $S^{r,n}$ such that $u_{j_0,\dots,j_n} = 0$, there exists a subset I of $\{0, \dots, n\}$ such that

$$\sum_{\alpha \in I} i_\alpha \geq \sum_{\alpha \in I} J_\alpha \Longrightarrow u_{j_0,\dots,j_n} = 0.$$

Let us choose r basis vectors of F. By decomposing them into a base of E union of bases of $E_0, \dots, E_n$, we obtain $n+1$ sets of r line vectors $l_1^\alpha, \dots, l_r^\alpha$, $0 \leq \alpha \leq n$, which are of r-tuples of scalars. By hypothesis, each time you choose j_0 vectors among $l_1^0, \ldots, l_r^0$, j_1 vectors among $l_1^1, \ldots, l_r^1$, ..., j_n vectors among $l_1^n, \ldots, l_r^n$, the determinant of the matrix thus formed is 0.

Choose $n+1$ integers $j'_0, \ldots, j'_n$ satisfying $0 \leq j'_0 \leq j_0$, $0 \leq j'_1 \leq j_1$, ..., $0 \leq j'_n \leq j_n$, not all null, such that each time we take j'_0 vectors among $l_1^0, \ldots, l_r^0$, j'_1 vectors among $l_1^1, \ldots, l_r^1$, ..., j'_n vectors among $l_1^n, \ldots, l_r^n$, the space generated by these $j'_0 + j'_1 + \cdots + j'_n$ vectors is of dimension $\leq j'_0 + j'_1 + \cdots + j'_n - 1$, and which are minimal for this property. In other words, for all integers $k_0, \ldots, k_n$ satisfying $0 \leq k_0 \leq j'_0$, $0 \leq k_1 \leq j'_1$, ..., $0 \leq k_n \leq j'_n$ and $k_0 + k_1 + \cdots + k_n = j'_0 j'_1 + \cdots + j'_n - 1$, it is possible to choose k_0 vectors among $l_1^0, \ldots, l_r^0$, k_1 vectors among $l_1^1, \ldots, l_r^1$, ..., k_n vectors among $l_1^n, \ldots, l_r^n$, such that the space generated by these $k_0 + k_1 + \cdots + k_n$ vectors is of dimension $k_0 + k_1 + \cdots + k_n$. It is then easy to satisfy that the space generated by such vectors is independent of the choice of $k_0, \ldots, k_n$ and of these vectors.

This means that if I denotes the nonempty subset of $\{0, \ldots, n\}$ composed of α such that $j'_\alpha \geq 1$, then the space generated by all vectors $l_1^\alpha, \ldots, l_r^\alpha$, $\alpha \in I$, is of dimension $j'_0 + j'_1 + \cdots + j'_n - 1$.

In consequence, we have for any tuple $(i_0, \ldots, i_n)$ of $S^{r,n}$ the implication

$$\sum_{\alpha \in I} i_\alpha \geq \sum_{\alpha \in I} j'_\alpha \Longrightarrow u_{i_0, \ldots, i_n} = 0$$

and all the more so

$$\sum_{\alpha \in I} i_\alpha \geq \sum_{\alpha \in I} j_\alpha \Longrightarrow u_{i_0, \ldots, i_n} = 0.$$

That's what we wanted. □

Proof of Lemma 1.8 of Section 1.3. Let then S be a entire convex polyhedron of $S^{r,n}$ and S' a face of S which is defined in S by an equation of the form

$$\sum_{\alpha \in I} i_\alpha = d_I \quad \text{with} \quad d_I = \min_{(i_0, \ldots, i_n) \in S} \left\{ \sum_{\alpha \in I} i_\alpha \right\}.$$

This is particularly the case if S' is a boundary of S.

All subspaces F of E which are in $\mathrm{Gr}_S^{r,n}$ have with E_I an intersection of fixed dimension d_I. We therefore have on $\mathrm{Gr}_S^{r,n}$ of a well-defined morphism

$$F \mapsto (F \cap E_I) \oplus F/(F \cap E_I) \subseteq E_I \oplus E/E_I = E.$$

It follows from the definitions and Lemma 2.1 Section 2.1 that this morphism sends $\mathrm{Gr}_S^{r,n}$ to $\mathrm{Gr}_{S'}^{r,n}$. And we deduce from the canonical isomorphisms

$$\Lambda^r F \cong \Lambda^{d_I}(F \cap E_I) \otimes \Lambda^{r-d_I}(F/F \cap E_I)$$

that it is induced by the restriction

$$(u_s)_{s \in S} \mapsto (u_s)_{s \in S'}.$$

Finally, for S' any face of S, there exists a sequence

$$S = S_0,\ S_1,\ \ldots,\ S_k = S'$$

of faces of S ranging from S to S' and each of which is a boundary of the previous one. Then the restriction $(u_s)_{s \in S} \mapsto (u_s)_{s \in S'}$ is written as a composite of restrictions at the boundaries; according to what we have already seen, it sends $\mathrm{Gr}_S^{r,n}$ to $\mathrm{Gr}_{S'}^{r,n}$. □

For $\underline{S}$ a family of entire convex tiling of $S^{r,n}$ such that the intersection of any two of them either a face of each of the two, or $\mathrm{Gr}^{r,n}_{\underline{S}}$ the closed subscheme of

$$\mathbb{G}_m \backslash \prod_{(i_0,\ldots,i_n)\in \cup_{S\in\underline{S}} S} \left[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\} \right]$$

of tuples $(u_{i_0,\ldots,i_n})$ such that, for any element S of $\underline{S}$, the restraint tuple

$$(u_{i_0,\ldots,i_n})_{(i_0,\ldots,i_n)\in S}$$

is in $\mathrm{Gr}^{r,n}_S$. When $\underline{S}$ is a convex entire tiling of $S^{r,n}$ and as announced in Section 1.3, we will show later on $\mathrm{Gr}^{r,n}_{\underline{S}}$ as the fiber of $\Omega^{r,n}$ above marked point $\alpha_{\underline{S}}$ of the orbit $\mathcal{A}_{\underline{S}}$ in $\mathcal{A}^{r,n}$.

If S is an entire convex polyhedron in the space

$$\left\{ (x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{0\le\alpha\le n} x_\alpha = r \right\},$$

denote

$$d_I^S = \min_{(x_0,\ldots,x_n)\in S} \left\{ \sum_{\alpha\in I} x_\alpha \right\}$$

the convex family of integers indexed by subsets I of $\{0,\ldots,n\}$ that defines it.

Then, according to Lemma 1.7 and 1.8 of Section 1.3, the schemes $\mathrm{Gr}^{r,n}_{\underline{S}}$ classifies the families (F_S) of subspaces of dimension r in $E = E_0 \oplus \cdots \oplus E_n$ indexed by faces of elements of $\underline{S}$ such that:

– for any S and any subset I of $\{0,\ldots,n\}$,

$$\dim(F_S \cap E_I) = d_I^S,$$

– for any S and any face S' of S defined by an equation of the form

$$\sum_{\alpha\in I} x_\alpha = d_I^S,$$

$$F_{S'} = (F_S \cap E_I) \oplus F_S/(F_S \cap E_I).$$

This leads to the introduction of the stacks $\mathrm{Vec}^{r,n}_{\underline{S}}$ classifying the families (F^S) of vector spaces of dimension r indexed by faces of elements of $\underline{S}$ and equipped with following supplementary structures:

– for any S and any subset I of $\{0,\ldots,n\}$, a subspace F_I^S of F^S of dimension d_I^S, with the condition that

$$F_I^S \cap F_J^S = F_{I\cap J}^S,\ \forall\ I, J,$$

and in particular

$$F_I^S \subseteq F_J^S$$

if $I \subseteq J$,

– for any S and any face S' of S defined by an equation of the form

$$\sum_{\alpha\in I} x_\alpha = d_I^S,$$

of isomorphisms

$$F_I^{S'} \cong F_I^S,\ F_J^{S'} \cong F^S/F_I^S,$$

where $J = \{0, \ldots, n\} - I$ and then $F^{S'} = F_I^{S'} \oplus F_J^{S'}$, with the condition that each $F_K^{S'} = F_{I\cap K}^{S'} \oplus F_{J\cap K}^{S'}$ is transformed to $F_{I\cap K}^{S} \oplus F_{I\cup(J\cap K)}^{S}/F_I^S$. We also ask that for any S and any face S' of S, the various isomorphisms between $F^{S'}$ and a certain graduation of F^S deduced from the previous ones by composition are merged.

For any $\underline{S}$, we then have a natural projection

$$\mathrm{Gr}_{\underline{S}}^{r,n} \to \mathrm{Vec}_{\underline{S}}^{r,n}.$$

3.2 Smoothness of schemes of glued graphs

Given $\underline{S}$ a convex entire tiling of $S^{r,n}$, we would like to build the scheme $\mathrm{Gr}_{\underline{S}}^{r,n}$ by gluing one by one the schemes $\mathrm{Gr}_S^{r,n}$ associated with tiles S of $\underline{S}$. To do this, we need to put an order relation on the set of tiles of $\underline{S}$. Let's do it this way:

Lemma 3.1. *The set of tiles of any convex entire tiling $\underline{S}$ of $S^{r,n}$ can be totally ordered so that, for any tile S of $\underline{S}$ and any subset $I \subsetneq \{0, \ldots, n\}$, we have:*

- *if $0 \in I$, the face of S of equation*

$$\sum_{\alpha \in I} x_\alpha = d_I^S$$

is contained in the union of boundary of $S^{r,n}$ of equation $x_0 = 0$ and of tiles $S' < S$ of $\underline{S}$,

- *if $0 \notin I$, the face of S of equation*

$$\sum_{\alpha \in I} x_\alpha = d_I^S$$

is contained in the union of boundaries of $S^{r,n}$ of equations

$$x_1 = 0, \ \ldots, \ x_n = 0$$

and of tiles $S' > S$ of $\underline{S}$.

Proof. By definition of convex entire tilings, there exists a convex function v on $S^{r,n}$ such that the tiles of $\underline{S}$ are the maximal entire tiles on which v is affine. If x is a nonzero vector of the space

$$\Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{0 \leq \alpha \leq n} x_\alpha = 0\Big\},$$

we have for any tile S of $\underline{S}$ of the slope $\frac{\partial v}{\partial x}(S)$ in the direction x of the affine map v restraint to S.

We can choose the vector $x = (x_0, \ldots, x_n)$ so that the slopes $\frac{\partial v}{\partial x}(S)$ are two by two distinct and that $x_0 > 0$, $x_1, \ldots, x_n < 0$ so that for any subset $I \subsetneq \{0, \ldots, n\}$ we have $\sum_{\alpha\in I} x_\alpha > 0$ if $0 \in I$ and $\sum_{\alpha \in I} x_\alpha < 0$ if $0 \notin I$.

Then the order relation defined by

$$S < S' \Longleftrightarrow \frac{\partial v}{\partial x}(S) < \frac{\partial v}{\partial x}(S')$$

answers the question asked. □

The convex entire tiling $\underline{S}$ of $S^{r,n}$ will now be provided with such a total order relation. We denote $\underline{S}_0$ the convex entire tiling induced by $\underline{S}$ on the boundary of $S^{r,n}$ of equation $x_0 = 0$ identified with $S^{r,n-1}$; this is a finite set of tiles of $S^{r,n-1}$. And for $0 \le k \le \#\underline{S}$, we denote $\underline{S}_k$ the set of first k tiles of $\underline{S}$ and those of the tiles of $\underline{S}_0$ which are not boundaries of these.

We have a commutative diagram

$$\begin{array}{ccccccccc} \mathrm{Gr}^{r,n}_{\underline{S}_0} & \longleftarrow & \mathrm{Gr}^{r,n}_{\underline{S}_1} & \longleftarrow \cdots \longleftarrow & \mathrm{Gr}^{r,n}_{\underline{S}_k} & \longleftarrow \cdots \longleftarrow & \mathrm{Gr}^{r,n}_{\underline{S}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Vec}^{r,n}_{\underline{S}_0} & \longleftarrow & \mathrm{Vec}^{r,n}_{\underline{S}_1} & \longleftarrow \cdots \longleftarrow & \mathrm{Vec}^{r,n}_{\underline{S}_k} & \longleftarrow \cdots \longleftarrow & \mathrm{Vec}^{r,n}_{\underline{S}} \end{array}$$

for which we are going to show:

Theorem 3.2. *With the notations above, the stacks* $\mathrm{Vec}^{r,n}_{\underline{S}_k}$, $0 \le k \le \#\underline{S}$, *are algebraic (in the sense of Artin), smooth and equidimensional and the schemes* $\mathrm{Gr}^{r,n}_{\underline{S}_k}$ *are smooth on* $\mathrm{Vec}^{r,n}_{\underline{S}_k}$*, thus smooth, and equidimensional.*

In addition, for $1 \le k \le \#\underline{S}$*, the morphisms* $\mathrm{Vec}^{r,n}_{\underline{S}_k} \to \mathrm{Vec}^{r,n}_{\underline{S}_k}$ *are smooth and if* S *is the* k*-th tile of* $\underline{S}$ *that we have added to* $\underline{S}_{k-1}$ *to form* $\underline{S}_k$*, we have*

$$\dim \mathrm{Gr}^{r,n}_{\underline{S}_k} - \dim \mathrm{Gr}^{r,n}_{\underline{S}_{k-1}} = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{n-\#I} \Big[(r - d^S_{\{0\}}) d^S_{\{0\}\cup I} - d^S_{\{1,\ldots,n\}} d^S_I \Big].$$

Proof. Proceeding by recurrence on n and k, we can assume have already shown that $\mathrm{Vec}^{r,n}_{\underline{S}_{k-1}}$ is algebraic, smooth and equidimensional and that $\mathrm{Gr}^{r,n}_{\underline{S}_{k-1}}$ is smooth on $\mathrm{Vec}^{r,n}_{\underline{S}_{k-1}}$, thus smooth, and equidimensional. Indeed, it is obvious for $\mathcal{V}ec^{r,1}_{\underline{S}_0}$ and $\mathrm{Gr}^{r,1}_{\underline{S}_0}$ and on the other hand, if $n \ge 2$, $\mathrm{Vec}^{r,n}_{\underline{S}_0}$ and $\mathrm{Gr}^{r,n}_{\underline{S}_0}$ are identified with $\mathrm{Vec}^{r,n-1}_{\underline{S}_0}$ and $\mathrm{Gr}^{r,n-1}_{\underline{S}_0}$.

Let then S be the tile of $\underline{S}$ that we have added to $\underline{S}_{k-1}$ to form $\underline{S}_k$ and let $\underline{S}'$ be the family of faces of S which is defined by equations of the form $\sum_{\alpha\in I} x_\alpha = d^S_I$ with $\{0\} \subseteq I \subsetneq \{0,\ldots,n\}$.

Then the stack $\mathrm{Vec}^{r,n}_{\underline{S}_k}$ is identified with 2-Cartesian product $\mathrm{Vec}^{r,n}_{\underline{S}_{k-1}} \times_{\mathrm{Vec}^{r,n}_{\underline{S}'}} \mathrm{Vec}^{r,n}_{S}$ and the scheme $\mathrm{Gr}^{r,n}_{\underline{S}_k}$ with Cartesian product $\mathrm{Gr}^{r,n}_{\underline{S}_{k-1}} \times_{\mathrm{Gr}^{r,n}_{\underline{S}'}} \mathrm{Gr}^{r,n}_{S}$.

The stack $\mathrm{Vec}^{r,n}_{S}$ classifies the vector spaces F of dimension r equipped with a family (F_I) of subspaces of dimension d^S_I indexed by subsets I of $\{0,\ldots,n\}$ such that, for any I, J, we have $F_I \cap F_J = F_{I\cap J}$ and in particular $F_I \subseteq F_J$ if $I \subseteq J$.

And the stack $\mathrm{Vec}^{r,n}_{\underline{S}'}$ classifies the spaces F' of dimension $r - d^S_{\{0\}}$ equipped with subspaces F'_I, $\{0\} \subseteq I \subseteq \{0,\ldots,n\}$, of dimension $d^S_I - d^S_{\{0\}}$ such that $F'_I \cap F'_J = F'_{I\cap J}$, $\forall\, I,\ J$, moreover the data of spaces F_I, $I \subsetneq \{0,\ldots,n\}$, of dimension d^S_I and of a coherent system of inclusions $F_I \hookrightarrow F_J$, $I \subseteq J$, with $F_I \cap F_J = F_{I\cap J}$, $I,\ J \subseteq K$, and of isomorphisms $F_I/F_{\{0\}} \cong F'_I$, $\{0\} \subseteq I \subsetneq \{0,\ldots,n\}$.

The morphism $\mathrm{Vec}^{r,n}_{S} \to \mathrm{Vec}^{r,n}_{\underline{S}'}$ consists of associating to spaces F equipped with F_I, $I \subseteq \{0,\ldots,n\}$, the spaces $F' = F/F_{\{0\}}$ equipped with $F'_I = F_I/F_{\{0\}}$, $\{0\} \subseteq I \subseteq \{0,\ldots,n\}$, moreover the sub-families $(F_I)_{I\subsetneq\{0,\ldots,n\}}$.

Let again $\mathrm{Hom}^{r,n}_{S}$ the stack classifying the families composed of a subspace $F_{\{0\}}$ of E_0 of dimension $d^S_{\{0\}}$, of a space F' of dimension $r - d^S_{\{0\}}$ equipped with subspaces F'_I, $\{0\} \subseteq I \subseteq \{0,\ldots,n\}$, as above and of a homomorphism $u : F' \to E_0/F_{\{0\}}$ such that for any $I \subseteq \{0,\dots,n\}$, $\dim(\operatorname{Ker} u \cap F'_{\{0\}\cup I}) = d_I^S$. By associating with any such family the space $F = \operatorname{Ker}(E_0 \oplus F' \to E_0/F_{\{0\}})$ equipped with subspaces

$$F_I = \operatorname{Ker}(E_0 \oplus F'_I \to E_0/F_{\{0\}}),\ \{0\} \subseteq I \subseteq \{0,\dots,n\}$$

and

$$F_I = \operatorname{Ker} u \cap F'_{\{0\}\cup I},\ I \subseteq \{1,\dots,n\},$$

we defines a morphism $\mathrm{Hom}_S^{r,n} \to \mathrm{Vec}_S^{r,n}$.

Likewise, let $\mathrm{Hom}_{\underline{S}'}^{r,n}$ the stack classifying the families composed of a subspace $F_{\{0\}}$ of E_0 of dimension $d^S_{\{0\}}$, of a space F' of dimension $r - d^S_{\{0\}}$ equipped with subspaces

$$F'_I,\ \{0\} \subseteq I \subseteq \{0,\dots,n\}$$

as above and of homomorphisms

$$u_I : F'_{\{0\}\cup I} \to E_0/F_{\{0\}},\ I \subsetneq \{0,\dots,n\},$$

whose kernels are of dimensions d_I^S and which are compatible with the inclusions $F'_{\{0\}\cup I} \subseteq F'_{\{0\}\cup J}$, $I \subseteq J$. Here again we have a morphism $\mathrm{Hom}_{\underline{S}'}^{r,n} \to \mathrm{Vec}_{\underline{S}'}^{r,n}$.

To restrict to $F'_{\{0\}\cup I}$, $I \subsetneq \{0,\dots,n\}$, the homomorphisms $u : F' \to E_0/F_{\{0\}}$ defines a morphism $\mathrm{Hom}_S^{r,n} \to \mathrm{Hom}_{\underline{S}'}^{r,n}$ above $\mathrm{Vec}_S^{r,n} \to \mathrm{Vec}_{\underline{S}'}^{r,n}$.

Remark that the data of a subspace F of $E = E_0 \oplus \cdots \oplus E_n$ is equivalent to those of subspaces $F \cap E_0$ of E_0 and $F/(F\cap E_0)$ of $E_1 \oplus \cdots \oplus E_n$ and of a homomorphism $j : F/(F\cap E_0) \to E_0/(F\cap E_0)$. Then we have natural morphisms

$$\mathrm{Gr}_S^{r,n} \to \mathrm{Hom}_S^{r,n}$$

and $\mathrm{Gr}_{\underline{S}'}^{r,n} \to \mathrm{Hom}_{\underline{S}'}^{r,n}$ and $\mathrm{Gr}_S^{r,n}$ is identified with $\mathrm{Gr}_{\underline{S}'}^{r,n} \times_{\mathrm{Hom}_{\underline{S}'}^{r,n}} \mathrm{Hom}_S^{r,n}$.

In sequence, $\mathrm{Gr}_{\underline{S}_k}^{r,n}$ is identified with $\mathrm{Gr}_{\underline{S}_{k-1}}^{r,n} \times_{\mathrm{Hom}_{\underline{S}'}^{r,n}} \mathrm{Hom}_S^{r,n}$ and we are led to prove the following lemma:

Lemma 3.3. *With these notations, we have:*

(i) *The stacks* $\mathrm{Vec}_{\underline{S}'}^{r,n}$ *and* $\mathrm{Vec}_S^{r,n}$ *are algebraic, smooth and equidimensional and the morphism* $\mathrm{Vec}_S^{r,n} \to \mathrm{Vec}_{\underline{S}'}^{r,n}$ *is smooth.*

(ii) *The schemes* $\mathrm{Hom}_{\underline{S}'}^{r,n}$ *and* $\mathrm{Hom}_S^{r,n}$ *are smooth on* $\mathrm{Vec}_{\underline{S}'}^{r,n}$ *and* $\mathrm{Vec}_S^{r,n}$, *thus smooth, and equidimensional. In addition,*

$$\dim \mathrm{Hom}_S^{r,n} - \dim \mathrm{Hom}_{\underline{S}'}^{r,n} = \sum_{I\subseteq\{1,\dots,n\}} (-1)^{n-\#I}\left[(r - d^S_{\{0\}})d^S_{\{0\}\cup I} - d^S_{\{1,\dots,n\}} d^S_I\right].$$

(iii) *The morphism*

$$\mathrm{Gr}_{\underline{S}_{k-1}}^{r,n} \to \mathrm{Hom}_{\underline{S}'}^{r,n} \times_{\mathrm{Vec}_{\underline{S}'}^{r,n}} \mathrm{Vec}_{\underline{S}_{k-1}}^{r,n}$$

is smooth.

Proof of Lemma 3.3. (i) Let us proceed by recurrence on the dimension n. First, let's build the stack $\mathrm{Vec}_{\underline{S}'}^{r,n}$. By hypothesis of recurrence, the stack of spaces F' of dimension $r - d^S_{\{0\}}$ equipped with

$$F'_I,\ \{0\} \subseteq I \subseteq \{0,\dots,n\}$$

is algebraic, smooth and equidimensional. It is the same, for $\{0\} \subseteq I \subseteq \{0, \ldots, n\}$ and $\#I = n$, of stack of spaces F_I of dimension d_I^S equipped with subspaces $F_{I,K}$, $K \subseteq I$, of dimensions d_K^S such that

$$F_{I,K} \cap F_{I,L} = F_{I,K\cap L}, \ \forall \ K, L \subseteq I.$$

Now the stack $\mathrm{Vec}_{\underline{S}'}^{r,n}$ is representable on the product of these stacks as that of families of isomorphisms

$$F_{I,K} \cong F_{J,K}, \{0\} \subseteq I, J \subsetneq \{0, \ldots, n\}, \ \#I = \#J = n, \ K \subseteq I \cap J$$

and

$$F_{I,K}/F_{I,\{0\}} \cong F'_K, \ \{0\} \subseteq K \subseteq I \subsetneq \{0, \ldots, n\}, \ \#I = n,$$

compatible with each other and with the different inclusions. It is therefore algebraic, smooth and equidimensional.

Let's see now the morphism $\mathrm{Vec}_S^{r,n} \to \mathrm{Vec}_{\underline{S}'}^{r,n}$. To reconstitute the space F with F_I, $I \subseteq \{0, \ldots, n\}$, choose F of dimension r and an embedding $F_{\{0\}} \hookrightarrow F$ and an isomorphism $F/F_{\{0\}} \cong F'$ then embeddings

$$F_I \hookrightarrow F, \ \{0\} \subseteq I \subsetneq \{0, \ldots, n\}, \ \#I = n,$$

which lifts $F_I/F_{\{0\}} \cong F'_I \hookrightarrow F'$ and that are compatible. We see that the stack $\mathrm{Vec}_S^{r,n}$ is algebraic, smooth on $\mathrm{Vec}_{\underline{S}'}^{r,n}$, thus smooth, and equidimensional.

(ii) Above $\mathrm{Vec}_{\underline{S}'}^{r,n}$, the stack $\mathrm{Hom}_{\underline{S}'}^{r,n}$ is is that of the embeddings $F_{\{0\}} \hookrightarrow E_0$ and of homomorphisms $u_I : F'_{\{0\}\cup I} \to E_0/F_{\{0\}}$, $I \subsetneq \{1, \ldots, n\}$, of kernel of $F_I \hookrightarrow F_{\{0\}\cup I}/F_{\{0\}} \cong F'_{\{0\}\cup I}$ and compatible with each other.

Likewise, above $\mathrm{Vec}_S^{r,n}$, the stack $\mathrm{Hom}_S^{r,n}$ is that of the embeddings $F_{\{0\}} \hookrightarrow E_0$ and of homomorphisms $u : F/F_{\{0\}} \to E_0/F_{\{0\}}$ pf kernel $F_{\{1,\ldots,n\}} \hookrightarrow F/F_{\{0\}}$.

Thus, $\mathrm{Hom}_{\underline{S}'}^{r,n}$ and $\mathrm{Hom}_S^{r,n}$ are smooth on $\mathrm{Vec}_{\underline{S}'}^{r,n}$ and $\mathrm{Vec}_S^{r,n}$, thus smooth, and equidimensional. More precisely, let's see how to go from $\mathrm{Hom}_{\underline{S}'}^{r,n}$ to $\mathrm{Hom}_S^{r,n}$. We must first choose a homomorphism $u : F/F_{\{0\}} \to E_0/F_{\{0\}}$ of rank $r - d_{\{0\}}^S - d_{\{1,\ldots,n\}}^S$,hence a difference in dimensions equal to

$$(r-d_{\{0\}}^S-d_{\{1,\ldots,n\}}^S)\Big(2(r-d_{\{0\}}^S)-(r-d_{\{0\}}^S-d_{\{1,\ldots,n\}}^S)\Big) = r^2-2rd_{\{0\}}^S+(d_{\{0\}}^S)^2-(d_{\{1,\ldots,n\}}^S)^2.$$

Then it is necessary to impose that for any $I \subsetneq \{0, \ldots, n\}$ we have $\mathrm{Ker}\, u_I \subseteq \mathrm{Ker}\, u$, hence a new difference in dimensions

$$\sum_{I\subsetneq\{0,\ldots,n\}} (-1)^{n-\#I}(r - d_{\{0\}}^S - d_{\{1,\ldots,n\}}^S)d_I^S.$$

Finally, it is necessary to impose that for any $I \subsetneq \{1, \ldots, n\}$, u_I coincides with u on $F'_{\{0\}\cup I}/\mathrm{Ker}\, u_I$, hence a final difference in dimensions

$$\sum_{I\subsetneq\{1,\ldots,n\}} (-1)^{n-\#I}(r - d_{\{0\}}^S)(d_{\{0\}\cup I}^S - d_{\{0\}}^S - d_I^S).$$

The total dimensional difference is therefore

$$\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{n-\#I}\Big[(r - d_{\{0\}}^S)d_{\{0\}\cup I}^S - (r - d_{\{0\}}^S)d_{\{0\}}^S - d_{\{1,\ldots,n\}}^S d_I^S\Big].$$

which is the announced formula since

$$\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{n-\#I} = 0.$$

(iii) Since $\mathrm{Hom}^{r,n}_{\underline{S}'}$ is smooth on $\mathrm{Vec}^{r,n}_{\underline{S}'}$, the fiber bundle product $\mathrm{Hom}^{r,n}_{\underline{S}'} \times_{\mathrm{Vec}^{r,n}_{\underline{S}'}} \mathrm{Vec}^{r,n}_{\underline{S}_{k-1}}$ is smooth on $\mathrm{Vec}^{r,n}_{\underline{S}_{k-1}}$ just like $\mathrm{Gr}^{r,n}_{\underline{S}_{k-1}}$. It suffices then to show that the morphisms induced by

$$\mathrm{Gr}^{r,n}_{\underline{S}_{k-1}} \to \mathrm{Hom}^{r,n}_{\underline{S}'} \times_{\mathrm{Vec}^{r,n}_{\underline{S}'}} \mathrm{Vec}^{r,n}_{\underline{S}_{k-1}}$$

above geometric points of $\mathrm{Vec}^{r,n}_{\underline{S}_{k-1}}$ are smooth. Now this is clear since they are restrictions to open subsets of certain projections between affine spaces. □

Final proof of Theorem 3.2. Given the recurrence assumption on $\mathrm{Vec}^{r,n}_{\underline{S}_{k-1}}$ and $\mathrm{Gr}^{r,n}_{\underline{S}_{k-1}}$, Theorem 3.2 is deduced from Lemma 3.3 according to the identification

$$\mathrm{Vec}^{r,n}_{\underline{S}_k} = \mathrm{Vec}^{r,n}_{\underline{S}_{k-1}} \times_{\mathrm{Vec}^{r,n}_{\underline{S}'}} \mathrm{Vec}^{r,n}_{S}$$

and

$$\mathrm{Gr}^{r,n}_{\underline{S}_k} = \mathrm{Gr}^{r,n}_{\underline{S}_{k-1}} \times_{\mathrm{Hom}^{r,n}_{\underline{S}'}} \mathrm{Hom}^{r,n}_{S}$$

and the factorization of morphism

$$\mathrm{Gr}^{r,n}_{\underline{S}_k} \to \mathrm{Vec}^{r,n}_{\underline{S}_k}$$

into

$$\mathrm{Hom}^{r,n}_{S} \times_{\mathrm{Vec}^{r,n}_{\underline{S}'}} \mathrm{Vec}^{r,n}_{\underline{S}_{k-1}} \to \mathrm{Vec}^{r,n}_{\underline{S}_k}.$$

□

□

3.3 Calculation of the dimension

Let us first introduce a convenient notation. If S is a entire convex polyhedron in the space

$$\{(x_0,\ldots,x_n)\in\mathbb{R}^{n+1} \mid \sum_{0\le\alpha\le n} x_\alpha = r\}$$

defined by the convex family of integers

$$(d_I^S = \min_{(x_0,\ldots,x_n)\in S} \{\sum_{\alpha\in I} x_\alpha\})_{I\subseteq\{0,\ldots,n\}},$$

we put

$$d^S = \sum_{I\subseteq\{0,\ldots,n\}} (-1)^{n+1-\#I} d_I^S.$$

Let's start with the following lemma:

Lemma 3.4. *For S a entire convex polyhedron which is not a tile i.e. having codimension ≥ 1, we have*

$$d^S = \sum_{I\subseteq\{0,\ldots,n\}} (-1)^{n+1-\#I} d_I^S = 0.$$

Proof. By hypothesis there exists a partition of $\{0,\dots,n\}$ into two nontrivial subsets J and K such that

$$r = d^S_{\{0,\dots,n\}} = d^S_J + d^S_K$$

and

$$\sum_{\alpha\in J} x_\alpha = d^S_J,\ \sum_{\alpha\in K} x_\alpha = d^S_K,\ \forall\,(x_0,\dots,x_n)\in S.$$

Then we deduce from Lemma 2.1 of Section 2.1 that for any subset I of $\{0,\dots,n\}$, we have

$$d^S_I = d^S_{I\cap J} + d^S_{I\cap K}$$

in addition

$$\#I = \#(I\cap J) + \#(I\cap K).$$

We obtain by consequence

$$\begin{aligned} d^S &= (-1)^{n+1}\sum_{I'\subseteq J}\sum_{I''\subseteq K}(-1)^{\#I'+\#I''}(d^S_{I'}+d^S_{I''}) \\ &= (-1)^{n+1}\sum_{I'\subseteq J}(-1)^{\#I'}d^S_{I'}\sum_{I''\subseteq K}(-1)^{\#I''} \\ &\quad + (-1)^{n+1}\sum_{I''\subseteq K}(-1)^{\#I''}d^S_{I''}\sum_{I'\subseteq J}(-1)^{\#I'} \end{aligned}$$

which is zero as announced. □

If S is a entire convex polyhedron in the space

$$\{(x_0,\dots,x_n)\in\mathbb{R}^{n+1} \mid \sum_{0\le\alpha\le n} x_\alpha = r\}$$

associated with a convex family of integers (d^S_I), we denote S_0 and S^0 the entier convex polyhedra of spaces

$$\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid \sum_{1\le\alpha\le n} x_\alpha = r - d^S_{\{0\}}\}$$

and

$$\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid \sum_{1\le\alpha\le n} x_\alpha = d^S_{\{1,\dots,n\}}\}$$

defined by

$$(x_1,\dots,x_n)\in S_0 \Longleftrightarrow (d^S_{\{0\}},x_1,\dots,x_n)\in S,$$

and

$$x_1,\dots,x_n)\in S^0 \Longleftrightarrow (r-d^S_{\{1,\dots,n\}},x_1,\dots,x_n)\in S.$$

Lemma 3.5. *For S a entire convex polyhedron defined by a convex family (d^S_I), the entire convex polyhedron associated to S_0 and S^0 are defined by the convex families*

$$d^{S_0}_I = d^S_{\{0\}\cup I} - d^S_{\{0\}},\ I\subseteq\{1,\dots,n\}$$

and

$$d^{S^0}_I = d^S_I,\ I\subseteq\{1,\dots,n\}.$$

In particular, we have

$$d^S = d^{S_0} - d^{S^0}.$$

Proof. The first assertion is resulted from Lemma 2.3 of Section 2.1. It leads to the second according to the definitions and the formula

$$\sum_{I\subseteq\{1,\ldots,n\}}(-1)^{\#I}=0.$$

□

From Lemma 3.4 and 3.5 we deduce that the invariants d^S attached to entier convex polyhedra S are additive.

Proposition 3.6. *Let $\underline{S}$ be a convex entire tiling of simplex $S^{r,n}$. Then if S runs through the set of tiles of $\underline{S}$, we have*

$$\sum_S d^S = d^{S^{r,n}} = r.$$

Proof. We proceed by recurrence on n. For $n=1$, the formula expresses the obvious fact that the sum of the lengths of any finite family of intervals constituting a partition of $[0,r]$ is r.

Suppose then $n\geq 2$ and the formula already established in rank $n-1$.

According to Lemma 3.5, we can write

$$\sum_S d^S=\sum_S(d^{S_0}-d^{S^0})$$

and according to Lemma 3.4 one can remember in this sum only S_0 and S^0 which are dimension $n-1$. Now any face of $\underline{S}$ of dimension $n-1$ is shared by two tiles of $\underline{S}$ exactly, unless it is on a boundary of $S^{r,n}$. After simplification we obtain

$$\sum_S d^S=\sum_{S,\ d^S_{\{0\}}=0} d^{S_0}.$$

Now, when S runs through the subset of tiles of $\underline{S}$ such that $d^S_{\{0\}}=0$, S_0 constitutes the convex entire tiling of $S^{r,n-1}$ induced by that of $S^{r,n}$ via the embedding

$$S^{r,n-1}\hookrightarrow S^{r,n}$$

$$(x_1,\ldots,x_n)\mapsto(0,x_1,\ldots,x_n)$$

We conclude according to the hypothesis of recurrence. □

The above combined results now allow us to deduce from Theorem 3.2 of the preceding section the corollary:

Corollary 3.7. *For any convex entire tiling $\underline{S}$ of $S^{r,n}$, the scheme of glued graph $\mathrm{Gr}^{r,n}_{\underline{S}}$ is smooth of constant dimension nr^2.*

Proof. Let $\underline{S}_0$ be the convex entire tiling induced by $\underline{S}$ on the boundary of $S^{r,n}$ of equation $x_0=0$ identified with $S^{r,n-1}$. According to Theorem 3.2, $\mathrm{Gr}^{r,n-1}_{\underline{S}_0}$ and $\mathrm{Gr}^{r,n}_{\underline{S}}$ are smooth and equidimensional and if S runs through the set of tiles of $\underline{S}$, we have

$$\begin{aligned}&\dim \mathrm{Gr}^{r,n}_{\underline{S}}-\dim \mathrm{Gr}^{r,n-1}_{\underline{S}_0}\\=&\sum_S\sum_{I\subseteq\{1,\ldots,n\}}(-1)^{n-\#I}\left[(r-d^S_{\{0\}})d^S_{\{0\}\cup I}-d^S_{\{1,\ldots,n\}}d^S_I\right].\end{aligned}$$

If $n = 1$, it is obvious that $\mathrm{Gr}^{r,0}_{\underline{S}_0}$ is of dimension 0 and if $n \geq 2$ we can proceed by recurrence and suppose already proved that $\mathrm{Gr}^{r,n-1}_{\underline{S}_0}$ is of dimension $(n-1)r^2$. It is remained to calculate the sum on the tiles S of $\underline{S}$ above.

According to Lemma 3.5, it is still written

$$\sum_S \left[(r - d^S_{\{0\}}) d^{S_0} - d^S_{\{1,\ldots,n\}} d^{S^0} \right]$$

and according to Lemma 3.4 we can retain in this sum only the S_0 and S^0 which are of dimension $n-1$. And these, unless they are on the boundary of $S^{r,n}$ of equation $x_0 = 0$, are shared by two tiles S, S' of $\underline{S}$ exactly and which satisfy

$$d^S_{\{1,\ldots,n\}} = r - d^{S'}_{\{0\}}, \quad d^{S^0} = d^{S'_0}.$$

After simplification, the sum reduces to

$$\sum_{S, d^S_{\{0\}} = 0} r d^{S_0}.$$

Now, when S runs through the subsets of tiles of $\underline{S}$ such that $d^S_{\{0\}} = 0$, S_0 runs through the entire convex tiling $\underline{S}_0$ of $\S^{r,n-1}$. According to Proposition 3.6, on conclude as wanted that the sum above is equal to r^2. □

Chapter 4

The properties of schemes of n-complete homomorphisms. Application to the Lang's isogeny

4.1 Projectivity of the representation morphisms

These are what we call the projections:

$$\Omega^{r,n} \to \mathbb{G}_m \Big\backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big],$$

$$\overline{\Omega}^{r,n} \to \prod_{(i_0,\dots,i_n)\in S^{r,n}} \mathbb{P}(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r).$$

By construction, their restrictions on open dense subsets

$$\Omega^{r,n}_{\emptyset} \cong (\mathrm{GL}_r^{n+1} \times \mathbb{G}_m^{S^{r,n}})/(\mathrm{GL}_r \times \mathbb{G}_m^{n+1})$$

and

$$\overline{\Omega}^{r,n}_{\emptyset} \cong \mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$$

are locally closed immersions, and on the other hand the second is the quotient of the first by the free actions of torus $\mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m$.

We will demonstrate here Theorem 1.5 of Section 1.2 which asserts that these two morphisms are projective.

It is enough to verify that the first one is and, as we already know that it is quasi-projective, that it satisfies the valuative criterion of properness. Thus, we have to see that any point of

$$\mathbb{G}_m \Big\backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

with values in a discrete valuation ring A and which generically is in $\Omega^{r,n}_{\emptyset}$ lifts (uniquely) at a point of $\Omega^{r,n}(A)$.

Referring to the construction of the toric variety $\mathcal{A}^{r,n}$, this means exactly that if u is a point of $\mathrm{GL}_r^{n+1} / \mathrm{GL}_r$ with values in a field K equipped with a discrete valuation v_K and

$$(u_{i_0,\dots,i_n})_{(i_0,\dots,i_n)\in S^{r,n}}$$

a tuple which is the representation in

$$\prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big],$$

then the map $v : S^{r,n} \to \mathbb{Z}$ which to any index $(i_0,\dots,i_n) \in S^{r,n}$ associates the minimum of valuations of coordinates of $u_{i_0,\dots,i_n} \in (\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}$ must be in the cone $\mathcal{C}^{r,n}$. In other words, we must show that for any affine map $l : S^{r,n} \to \mathbb{R}$ satisfying $l \leq v$, the set

$$\{s \in S^{r,n} \mid l(s) = v(s)\}$$

is an entire convex polyhedron as long as it is not empty.

As v takes its values in $\mathbb{Z}$, we can suppose that l takes his values in $\mathbb{Q}$ and also in $\mathbb{Z}$, if we replace the defining field K of u by a sufficiently ramified finite extension. In these conditions, there exists an element $\lambda = (\lambda_0, \ldots, \lambda_n) \in (K^\times)^{n+1}$ whose image

$$(\lambda_{i_0,\ldots,i_n} = \lambda_0 \lambda_1^{-i_1} \cdots \lambda_n^{-i_n})_{(i_0,\ldots,i_n)\in S^{r,n}}$$

in $(K^\times)^{S^{r,n}}$ satisfies

$$l(i_0, \ldots, i_n) = v_K(\lambda_{i_0,\ldots,i_n}), \ \forall \ (i_0, \ldots, i_n) \in S^{r,n}.$$

Then the tuple

$$(\lambda^{-1}_{i_0,\ldots,i_n} u_{i_0,\ldots,i_n})_{(i_0,\ldots,i_n)\in S^{r,n}}$$

defines a point of scheme

$$\Big(\prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r\Big) - \{0\}$$

with values in the valuation ring A of K and it represents the unique point of $\mathrm{Gr}^{r,n}(A)$ which extends the point $\lambda^{-1}u \in (\mathrm{GL}_r^{n+1} / \mathrm{GL}_r)(K)$. We conclude according to Lemma 1.7 of Section 1.3.

4.2 Smoothness of stratification morphism. Identification of its fibers

Here we want to prove Theorem 1.6(i) and Theorem 1.9 of Section 1.3.

For any convex entire tiling $\underline{S}$ of $S^{r,n}$, we have already constructed the closed subscheme $\mathrm{Gr}^{r,n}_{\underline{S}}$ of

$$\mathbb{G}_m \Big\backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

which is smooth of dimension nr^2 and admits the modular characterization of Theorem 1.9 of Section 1.3. We remark that the sub-torus $\mathbb{G}_m^{\underline{S}}$ of $\mathbb{G}_m^{S^{r,n}}$ stabilizer of marked point $\alpha_{\underline{S}}$ of the orbit $\mathcal{A}_{\underline{S}}$ in $\mathcal{A}^{r,n}$ preserves $\mathrm{Gr}^{r,n}_{\underline{S}}$; this results from the fact that for any tile S of $\underline{S}$, the morphism $\mathbb{G}_m^{\underline{S}} \hookrightarrow \mathbb{G}_m^{S^{r,n}} \to \mathbb{G}_m^{S}$ is factored through $\mathbb{G}_m^{n+1} \hookrightarrow \mathbb{G}_m^{S^{r,n}} \to \mathbb{G}_m^{S}$. If in consequence we send $\mathrm{Gr}^{r,n}_{\underline{S}}$ to $\alpha_{\underline{S}}$, we define a locally closed immersion of $\mathbb{G}_m^{\underline{S}} \backslash (\mathbb{G}_m^{S^{r,n}} \times \mathrm{Gr}^{r,n}_{\underline{S}})$ in

$$\mathbb{G}_m \Big\backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big] \times \mathcal{A}^{r,n}.$$

Lemma 4.1. *Let A be a integral local ring and*

$$\mathrm{Spec}\, A \to \mathbb{G}_m \Big\backslash \prod_{(i_0,\ldots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big] \times \mathcal{A}^{r,n}$$

be a morphism which generically is factored through $\mathbb{G}_m^{\underline{S}} \backslash (\mathbb{G}_m^{S^{r,n}} \times \mathrm{Gr}^{r,n}_{\underline{S}})$.

We suppose that the closed point of $\mathrm{Spec}\, A$ *is sent in $\mathcal{A}^{r,n}$ to a marked point $\alpha_{\underline{S}'}$ associated to a convex entire tiling $\underline{S}'$ which necessarily refining $\underline{S}$, so that the preimage of $\alpha_{\underline{S}'}$ in* $\mathrm{Spec}\, A$ *is a closed subset defined by an ideal I of A.*

Then $\mathrm{Spec}(A/I)$ *is sent in the closed subscheme* $\mathrm{Gr}^{r,n}_{\underline{S}'}$.

Proof. Denote K the fraction field of A. As the exact sequence

$$1 \to \mathbb{G}_m^{\underline{S}} \to \mathbb{G}_m^{S^{r,n}} \to \mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m^{\underline{S}} \to 1$$

is split, the induced point

$$\operatorname{Spec} K \to \mathbb{G}_m \Big\backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big]$$

can can be represented by a tuple of the form $(u_{i_0,\dots,i_n}) = (\lambda_{i_0,\dots,i_n} g_{i_0,\dots,i_n})$ where $\lambda_{i_0,\dots,i_n}$ are scalars in $K^\times$ and $g_{i_0,\dots,i_n}$ are elements of

$$(\Lambda^{i_0}K^r \otimes \cdots \otimes \Lambda^{i_n}K^r) - \{0\}$$

which defines an element of $\mathrm{Gr}_{\underline{S}}^{r,n}(K)$. And as this point extends to $\operatorname{Spec} A$, we can also suppose (by imposing, for example $u_{r,0,\dots,0} = 1$) that each $u_{i_0,\dots,i_n}$ has coefficients in A, at least one being invertible.

Let then S' be a tile of $\underline{S}'$ and S the tile of $\underline{S}$ which contains it. According to Lemma 1.2(iv) of Section 1.1, S' contains a generating family. If we modify tuples $(\lambda_s)_{s\in S^{r,n}}$ and $(g_s)_{s\in S^{r,n}}$ by an element of $(K^\times)^{n+1} \subset (K^\times)^{S^{r,n}}$, we can suppose that $\lambda_s = 1$ in the s of chosen generating family. By construction of the toric variety $\mathcal{A}^{r,n}$ this means that the λ_s^{-1}, $s \in S^{r,n}$, are in A, that they are congruent to 1 modulo I when $s \in S'$ and they are in I when $s \in S^{r,n} - S'$. In consequence, the tuple $\big((u_s)_{s\in S'}, (0)_{s\in S^{r,n}-S'}\big)$ is congruent module I on tuple $\big((g_s)_{s\in S'}, (0)_{s\in S^{r,n}-S'}\big)$ thus defines a point with values in A/I of the stratum $\mathrm{Gr}_{S'}^{r,n}$ of the Grassmannian $\mathrm{Gr}^{r,n}$.

This is true for any tile S' of $\underline{S}'$, we are done. ∎

According to Lemma 4.1, there exists a (unique) closed subscheme Ω of

$$\mathbb{G}_m \Big\backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big] \times \mathcal{A}^{r,n}$$

which is invariant by the action of torus $\mathbb{G}_m^{S^{r,n}}$ and whose fiber above each marked point $\alpha_{\underline{S}}$ of $\mathcal{A}^{r,n}$ is equal to $\mathrm{Gr}_{\underline{S}}^{r,n}$. Thus, all fibers of this scheme above points of $\mathcal{A}^{r,n}$ are smooth of dimension nr^2; in addition, it merges above the dense orbit $\mathcal{A}_\emptyset^{r,n}$ of $\mathcal{A}^{r,n}$ with the open dense subset $\Omega_\emptyset^{r,n}$ of scheme $\Omega^{r,n}$ then it contains $\Omega^{r,n}$ as closed subscheme.

We will have proven Theorem 1.6 (i) and Theorem 1.9 of Section 1.1 if we proved that this scheme ω is equal to $\Omega^{r,n}$ and it is smooth of relative dimension nr^2 above $\mathcal{A}^{r,n}$. But this follows from the following lemma:

Lemma 4.2. *Let $\underline{S}$ be a convex entire tiling of $S^{r,n}$ and u a closed point of scheme $\mathrm{Gr}_{\underline{S}}^{r,n}$.*

Then there exists a locally closed subscheme of

$$\mathbb{G}_m \Big\backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \cdots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big] \times \mathcal{A}^{r,n},$$

contains u, invariant by the action of torus $\mathbb{G}_m^{S^{r,n}}$, smooth of relative dimension nr^2 above $\mathcal{A}^{r,n}$ and whose fiber above marked point $\alpha_\emptyset = 1$ of the dense orbit $\mathcal{A}_\emptyset^{r,n} = \mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m^{n+1}$ of $\mathcal{A}^{r,n}$ is an open subset of $\mathrm{Gr}_\emptyset^{r,n} = \mathrm{GL}_r^{n+1}/\mathrm{GL}_r$.

Proof. Let $\operatorname{Spec} A$ be the affine scheme projective limit of affine open subschemes of

$$\mathbb{G}_m \Big\backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \dots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big] \times \mathcal{A}^{r,n}$$

which contains u and are invariant by the action of torus $\mathbb{G}_m^{S^{r,n}}$. Thus A is an integral ring whose field of fractions is identified with the field of functions of

$$\mathbb{G}_m \backslash \Big(\Lambda^r(\mathbb{A}^r)^{n+1} - \{0\}\Big) \times \mathcal{A}^{r,n};$$

it is invariant by the action of $\mathbb{G}_m^{S^{r,n}}$ and its finished points on $\mathcal{A}^{r,n}$ are the transforms of u by this action.

Let I be the ideal of A which defines the closed subscheme of $\operatorname{Spec} A$ induced by Ω. This ideal is generated by the subset $\mathscr{I}$ of its elements on which $\mathbb{G}_m^{S^{r,n}}$ acts by a character.

Recall on the other hand that the point u consists of a family of closed points u_S in stratum $\mathrm{Gr}_S^{r,n}$ of the Grassmannian $\mathrm{Gr}^{r,n}$. where S runs through the finite set of tiles of $\underline{S}$ and of their faces. Any point g of the fiber of $\operatorname{Spec} A/I$ above $\alpha_\emptyset$ is a point of $\mathrm{Gr}_\emptyset^{r,n}$; by construction of A, its orbit under the action of $\mathbb{G}_m^{S^{r,n}}$ contains in its closure the point u, and thus g, as a point of $\mathrm{Gr}_\emptyset^{r,n} \hookrightarrow \mathrm{Gr}^{r,n}$, contains in its closure at least one of the points u_S. The ideal I is again defined by the subset $\mathscr{I}'$ of $\mathscr{I}$ of its elements which are well defined in the neighborhood of the points u_S.

The codimension in

$$\mathbb{G}_m \backslash \Big(\Lambda^r(\mathbb{A}^r)^{n+1} - \{0\}\Big)$$

of smooth schemes $\mathrm{Gr}^{r,n}$ and $\mathrm{Gr}_{\underline{S}}^{r,n}$ is equal to $C^r_{(n+1)r} - 1 - nr^2$. It is thus possible to choose $C^r_{(n+1)r} - 1 - nr^2$ elements in $\mathscr{I}'$ whose relative differentials on $\mathcal{A}^{r,n}$ are linearly independent at the point u (so that it defines a closed subscheme of $\operatorname{Spec} A$ which is formally smooth of relative dimension nr^2 on $\mathcal{A}^{r,n}$) and at all points u_S (so that above $\alpha_\emptyset$ it generates the ideal I).

Then these $C^r_{(n+1)r} - 1 - nr^2$ equations define in the invariant open neiborhoods by $\mathbb{G}_m^{S^{r,n}}$ and sufficiently small of u in

$$\mathbb{G}_m \Big\backslash \prod_{(i_0,\dots,i_n)\in S^{r,n}} \Big[(\Lambda^{i_0}\mathbb{A}^r \otimes \dots \otimes \Lambda^{i_n}\mathbb{A}^r) - \{0\}\Big] \times \mathcal{A}^{r,n}$$

of closed subschemes that answer the question posed. □

4.3 Smoothness of morphisms of restrictions to faces

Let's give proof of Theorem 1.6(ii) of Section 1.3.

By functoriality, it is sufficient to prove that if

$$\iota : \{0,\dots,n-1\} \to \{0,\dots,n\}$$

is the injective map

$$\alpha \mapsto \alpha + 1,$$

then the induced map

$$\Omega^{r,n} \to \Omega^{r,n-1} \times_{\mathcal{A}^{r,n-1}} \mathcal{A}^{r,n}$$

is smooth of relative dimension r^2.

According to Theorem 1.6(i) of Section 1.3, the schemes $\Omega^{r,n}$ and $\Omega^{r,n-1} \times_{\mathcal{A}^{r,n-1}} \mathcal{A}^{r,n}$ are smooth of relative dimensions respectively nr^2 and $(n-1)r^2$ on $\mathcal{A}^{r,n}$, thus it suffices to show that the induced morphisms between fibers above points of $\mathcal{A}^{r,n}$ are smooth. In other words, it is necessary to see that if $\underline{S}$ is an entire convex tiling of the simplex

$$S^{r,n} = \{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1} \mid i_0 + \cdots + i_n = r\}$$

and $\underline{S}_0$ the induced convex entire tiling of boundary of equation $i_0 = 0$ identified with $S^{r,n-1}$, then the morphism of restriction to this boundary between schemes of glued graphs $\mathrm{Gr}_{\underline{S}}^{r,n} \to \mathrm{Gr}_{\underline{S}_0}^{r,n-1}$ is smooth.

Now, with the notations of Section 3.2, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_{\underline{S}}^{r,n} & \longrightarrow & \mathrm{Gr}_{\underline{S}_0}^{r,n-1} \\ \downarrow & & \downarrow \\ \mathrm{Vec}_{\underline{S}}^{r,n} & \longrightarrow & \mathrm{Vec}_{\underline{S}_0}^{r,n-1} \end{array}$$

and according to Theorem 3.2 of this section, we already know that schemes $\mathrm{Gr}_{\underline{S}}^{r,n}$ and $\mathrm{Gr}_{\underline{S}_0}^{r,n-1}$ are smooth on the stacks $\mathrm{Vec}_{\underline{S}}^{r,n}$ and $\mathrm{Vec}_{\underline{S}_0}^{r,n-1}$ and that the morphism $\mathrm{Vec}_{\underline{S}}^{r,n} \to \mathrm{Vec}_{\underline{S}_0}^{r,n-1}$ is smooth. Thus, we are reduced to prove that the morphism

$$\mathrm{Gr}_{\underline{S}}^{r,n} \to \mathrm{Gr}_{\underline{S}_0}^{r,n-1} \times_{\mathrm{Vec}_{\underline{S}_0}^{r,n-1}} \mathrm{Vec}_{\underline{S}}^{r,n}$$

is smooth and for that the induced morphisms between fibers above the points of $\mathrm{Vec}_{\underline{S}}^{r,n}$ are smooth. Now, above a point of $\mathrm{Vec}_{\underline{S}}^{r,n}$ with values in a field K, the fibers of $\mathrm{Gr}_{\underline{S}}^{r,n}$ and $\mathrm{Gr}_{\underline{S}_0}^{r,n-1} \times_{\mathrm{Vec}_{\underline{S}_0}^{r,n-1}} \mathrm{Vec}_{\underline{S}}^{r,n}$ are representable by two open subsets and two vector spaces of finite dimension on K and the morphism from one to the other is induced by a linear map between these spaces which is surjective; indeed, the canonical splitting of the exact sequence

$$0 \to E_0 \to E \to E_{\{1,\ldots,n\}} \to 0$$

defines for it a section.

This finishes the proof of Theorem 1.6(ii) of Section 1.3 and thus of all the results stated in Chapter 1.

4.4 Application to Lang's isogeny

In this final section, we take for base a finite field $\mathbb{F}_q$ with q elements. Thus all schemes on this base are equipped with Frobenius morphism of elevation to power q, which we denote as τ.

We will construct a projective compactification $\overline{\Omega}^{r,\tau}$ of PGL_r equipped with two morphisms to the compactification $\overline{\Omega}^{r,1}$ of $\mathrm{PGL}_r^2 \,/\, \mathrm{PGL}_r \cong \mathrm{PGL}_r$ included in two commutative diagrams:

$$\begin{array}{ccccc} g & & \mathrm{PGL}_r & \hookrightarrow & \overline{\Omega}^{r,\tau} \\ \downarrow & & \downarrow & & \\ g & & \mathrm{PGL}_r & \hookrightarrow & \overline{\Omega}^{r,1} \end{array}$$

$$\begin{array}{ccc} g & & \mathrm{PGL}_r \hookrightarrow \overline{\Omega}^{r,\tau} \\ \downarrow & & \downarrow \\ t(g)^{-1} \circ g & & \mathrm{PGL}_r \hookrightarrow \overline{\Omega}^{r,1} \end{array}$$

In this sense, $\overline{\Omega}^{r,\tau}$ will perform a compactification of the Lang isogeny $g \mapsto \tau(g)^{-1} \circ g$ in PGL_r.

As in Chapter 1, we first build a toric variety $\mathcal{A}^{r,\tau}$ which will serve as a parametrization scheme.

Recall that in the real vector space of functions

$$v : S^{r,2} = \{(i_0, i_1, i_2) \in \mathbb{N}^3 \mid i_0 + i_1 + i_2 = r\} \to \mathbb{R},$$

we have defined the cone $\mathcal{C}^{r,2}$ which corresponds to the toric variety $\mathcal{A}^{r,2}$. It is written naturally as the disjoint union of convex cones $\mathcal{C}^{r,2}_{\underline{S}}$ indexed by the convex entire tilings of triangle $S^{r,2}$. Let then $\mathcal{C}^{r,\tau}$ be the intersection of $\mathcal{C}^{r,2}$ with the subspace of functions $v : S^{r,2} \to \mathbb{R}$ such that for any $i_1, i_2 \in \mathbb{N}^2$, $i_1 + i_2 = r$, we have $v(0, i_1, i_2) = qv(i_1, 0, i_2)$. And let $\mathcal{C}^{r,\tau}_{\underline{S}}$ be the intersection convex cones of $\mathcal{C}^{r,2}_{\underline{S}}$ with this same subspace; we call q-convex entire tilings of $S^{r,2}$ those entire tilings $\underline{S}$ for which $\mathcal{C}^{r,\tau}_{\underline{S}}$ is not empty. The trivial tiling $\emptyset$ of $S^{r,2}$ is q-convex and $\mathcal{C}^{r,\tau}_{\emptyset}$ is the subspace of functions of the form $S^{r,2} \to \mathbb{R} : (i_0, i_1, i_2) \mapsto ai_0 + qai_1$ with $a \in \mathbb{R}$.

Of course, $\mathcal{C}^{r,\tau}$ is the disjoint union of $\mathcal{C}^{r,\tau}_{\underline{S}}$ and this satisfying the same properties of that in Proposition 1.3 in Section 1.1. The general theory showed in [Kem+73][§2] thus allows to associate to $\mathcal{C}^{r,\tau}$ a normal toric variety $\mathcal{A}^{r,\tau}$ of torus $\mathcal{A}^{r,\tau}_{\emptyset} = \mathbb{G}_m^{S^{r,\tau}}/\mathbb{G}_m$ where

$$S^{r,\tau} = \{(i_0, i_1, i_2) \in \mathbb{N}^3 \mid i_0 + i_1 + i_2 = r \text{ and } i_0 \neq 0\} \subsetneq S^{r,2}$$

and $\mathbb{G}_m$ is embedded in the torus $\mathbb{G}_m^{S^{r,\tau}}$ by $\lambda \mapsto (\lambda_{i_0,i_1,i_2} = \lambda^{i_0 + qi_1})$. The orbits in $\mathcal{A}^{r,\tau}$ are locally closed subschemes indexed naturally by the q-convex entire tilings $\underline{S}$ of $S^{r,2}$; we denote it as $\mathcal{A}^{r,\tau}_{\underline{S}}$. The closure of an orbit $\mathcal{A}^{r,\tau}_{\underline{S}}$ is the union of $\mathcal{A}^{r,\tau}_{\underline{S'}}$ for S' refining $\underline{S}$.

The embedding

$$\mathbb{G}_m^{S^{r,\tau}} \hookrightarrow \mathbb{G}_m^{S^{r,2}} : (\lambda_{i_0,i_1,i_2})_{i_0 \neq 0} \mapsto (\lambda_{i_0,i_1,i_2})$$

where $\lambda_{0,i_1,i_2} = \lambda^q_{i_1,0,i_2}$ if $i_1 \neq 0$ and $\lambda_{0,0,r} = 1$ induces an embedding $\mathbb{G}_m^{S^{r,\tau}}/\mathbb{G}_m \hookrightarrow \mathbb{G}_m^{S^{r,2}}/\mathbb{G}_m^3$ which in turn extends into an equivariant closed immersion $\mathcal{A}^{r,\tau} \hookrightarrow \mathcal{A}^{r,2}$. Each orbit $\mathcal{A}^{r,\tau}_{\underline{S}}$ in $\mathcal{A}^{r,\tau}$ has a marked point which is sent by this immersion to the marked point $\alpha_{\underline{S}}$ of $\mathcal{A}^{r,2}_{\underline{S}}$ and that therefore we can again denote as $\alpha_{\underline{S}}$.

The three strictly increasing maps $\{0, 1\} \to \{0, 1, 2\}$ induces three injections

$$\begin{aligned} S^{r,1} = \{(i_0, i_1) \in \mathbb{N}^2 \mid i_0 + i_1 = r\} &\to S^{r,2} \\ (i_0, i_1) &\mapsto (0, i_0, i_1),\ (i_0, 0, i_1),\ (i_0, i_1, 0) \end{aligned}$$

then three equivariant morphisms $p_0, p_1, p_2 : \mathcal{A}^{r,2} \to \mathcal{A}^{r,1}$ thus that three other $p_0, p_1, p_2 : \Omega^{r,2} \to \Omega^{r,1}$ above these. And $\mathcal{A}^{r,\tau}$ has been constructed in such a way that the three morphisms $p_0, p_1, p_2 : \mathcal{A}^{r,\tau} \to \mathcal{A}^{r,1}$ induced via the closed immersion $\mathcal{A}^{r,\tau} \hookrightarrow \mathcal{A}^{r,2}$ satisfying $p_0 = \tau \circ p_1$.

Theorem 4.3. *Let $\Omega^{r,\tau}$ be the closed subscheme of $\Omega^{r,2} \times_{\mathcal{A}^{r,2}} \mathcal{A}^{r,\tau}$ defined by the equation $p_0 = \tau \circ p_1$ in $\Omega^{r,1}$. Then:*

(i) *The scheme* $\Omega^{r,\tau}$ *is equipped with a free action of torus* $\mathbb{G}_m^{S^{r,\tau}}$ *and with two equivariant morphisms*

$$p_1, p_2 : \Omega^{r,\tau} \to \Omega^{r,1}$$

above

$$p_1, p_2 : \mathcal{A}^{r,\tau} \to \mathcal{A}^{r,1}$$

(ii) *The morphism* $\Omega^{r,\tau} \to \mathcal{A}^{r,\tau}$ *is equivariant and smooth of relative dimension* r^2.

Proof. (i) is obvious.

(ii) The smoothness results from the fact that, according to Theorem 1.6 of Section 1.3, the scheme $\Omega^{r,2}$ is smooth of relative dimension $2r^2$ on $\mathcal{A}^{r,2}$ and the morphism $p_0 : \Omega^{r,2} \to \Omega^{r,1} \times_{\mathcal{A}^{r,1}} \mathcal{A}^{r,2}$ is smooth of relative dimension r^2. □

If $\underline{S}$ is a q-convex entire tiling of triangle

$$S^{r,2} = \{(i_0, i_1, i_2) \in \mathbb{N}^3 \mid i_0 + i_1 + i_2 = r\},$$

we denote $\mathrm{Gr}_{\underline{S}}^{r,\tau}$ the fiber of $\Omega^{r,\tau}$ above marked point $\alpha_{\underline{S}}$ of the orbit $\mathcal{A}_{\underline{S}}^{r,\tau}$ in $\mathcal{A}^{r,\tau}$. As the tiling $\underline{S}$ is q-convex, it induces the same tiling $\underline{S}_0$ on the sides of equations $i_0 = 0$ and $i_1 = 0$ identified with $S^{r,1}$. Then $\mathrm{Gr}_{\underline{S}}^{r,\tau}$ is the closed subspace of $\mathrm{Gr}_{\underline{S}}^{r,2}$ defined by the equation $p_0 = \tau \circ p_1$ in $\mathrm{Gr}_{\underline{S}_0}^{r,1}$. Thus $\mathrm{Gr}_{\underline{S}}^{r,\tau}$ admits a modular interpretation deduced from cells of $\mathrm{Gr}_{\underline{S}}^{r,2}$, $\mathrm{Gr}_{\underline{S}}^{r,1}$ and $p_0, p_1 : \mathrm{Gr}_{\underline{S}}^{r,2} \to \mathrm{Gr}_{\underline{S}}^{r,1}$.

In particular, when $\underline{S}$ is the trivial tiling $\emptyset$, $\mathrm{Gr}_{\underline{S}_0}^{r,1} = \mathrm{Gr}_{\emptyset}^{r,1}$ is identified with $\mathrm{GL}_r^2 / \mathrm{GL}_r \cong \mathrm{GL}_r$ and $\mathrm{Gr}_{\underline{S}}^{r,\tau} = \mathrm{Gr}_{\emptyset}^{r,\tau}$ is identified with closed subscheme

$$\{(g_0, g_1, g_2) \in \mathrm{GL}_r^3 \mid g_1 = g_0 \circ g_2 \text{ and } g_0 = \tau(g_1)\}$$

so that $p_1 : (g_0, g_1, g_2) \mapsto g_1$ is an isomorphism and that $p_2 \circ p_1^{-1} : g_1 \mapsto \tau(g_1)^{-1} \circ g_1$ is the Lang's isogeny.

Finally, the quotient $\overline{\Omega}^{r,\tau}$ of $\Omega^{r,\tau}$ by the free action of torus $\mathbb{G}_m^{S^{r,\tau}}$ is a closed subscheme of $\overline{\Omega}^{r,2}$ thus a projective scheme. It contains PGL_r as open dense subset and it is equipped with two morphisms $p_1, p_2 : \overline{\Omega}^{r,2} \to \overline{\Omega}^{r,1}$ which realizes a compactification of the Lang's isogeny as announced at the beginning of this section.

Erratum

In this paper, we have constructed for any rank r equivariant compactifications $\overline{\Omega}^{r,n}$ of quotients $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$, $n \geq 1$. It is equipped with natural morphisms on "toric stacks" $\mathcal{A}^{r,n}/\mathcal{A}_\emptyset^{r,n}$ (quotients of toric varieties $\mathcal{A}^{r,n}$ par their tori $\mathcal{A}_\emptyset^{r,n} = \mathbb{G}_m^{S^{r,n}}/\mathbb{G}_m^{n+1}$) whose corresponding points of "convex entire" tilings of entire simplex

$$S^{r,n} = \{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1} \mid i_0 + \cdots + i_n = r\}$$

and their strata i.e. their fibers above these points admit a modular description in terms of corresponding tilings.

It was also claimed (it was Theorem 1.6 of Section 1.3) that for any rank r and any power $n \geq 1$, the morphism of structure $\overline{\Omega}^{r,n} \to \mathcal{A}^{r,n}\mathcal{A}_\emptyset^{r,n}$ was smooth and that for any injective map

$$\iota : \{0, \ldots, p\} \to \{0, \ldots, n\},$$

the induced morphism "of face"

$$\overline{\Omega}^{r,n} \to \overline{\Omega}^{r,p} \times_{\mathcal{A}^{r,p}/\mathcal{A}_\emptyset^{r,p}} \mathcal{A}^{r,n}/\mathcal{A}_\emptyset^{r,n}$$

was equally smooth.

But while preparing a lecture he was giving at the Institut Henri Poincaré, the author realized in the first days of June 2000 that his "proof" was wrong in general and that the case $n = 3$, $r = 4$ provides a counterexample to the smoothness statement: the projective compactification $\overline{\Omega}^{4,3}$ of $(\mathrm{PGL}_4)^4/\mathrm{PGL}_4$ is not even flat on $\mathcal{A}^{4,3}/\mathcal{A}_\emptyset^{4,3}$. Here are the series of cases where the smoothness statement is true and proven:

- When $n = 1$: this is the particular case $\mathrm{PGL}_r^2 / \mathrm{PGL}_r$ of "miraculous" compactifications of De Concini and Procesi.
- When $n = 2$: this is the first work of the author in this field, it was the subject of the pre-publication which contains a complete and correct demonstration.
- When r=2.

All this is detailed in a text that can be found on the electronic server of the IHES preprints (March 2001).

In the general case, the fault in the study of singularities of $\overline{\Omega}^{r,n}$ is located in the "proof" of Lemma 3.3 of Section 3.2. Also false are all the general statements of smoothness that depended on it, namely Theorem 3.2 and Lemma 3.3 of Section 3.2, Corollary 3.7 of Section 3.3, Lemma 2 of Section 4.2 and Theorem 1.6 of Section 1.3.

On the other hand, all the other statements concerning the construction of $\mathcal{A}^{r,n}$ and $\overline{\Omega}^{r,n}$ and on their projective, functorial and modular properties are correct.

Let us recall finally that the author had been led to the problem of compactification of $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$ by wanting to resolve the singularities of the stacks of Drinfeld shtuka of rank r with level structures. For arbitrary multiplicities, we have to compactify the adiagonal powers P^2/P and P^3/P of parabolic subgroups P, which is almost the same problem as compacting the $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$ in general. The author had thought to solve it in the preprint by giving a variant of his constructions of $\overline{\Omega}^{r,n}$; Here again, the constructions and all statements of projective, functorial and modular properties are correct but all statements of smoothness are false.

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Abstract. Some equivariant compactifications of the quotients $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$ are constructed. Each one is decomposed into locally closed strata which are smooth, are indexed by the entier convex pavings of the simplex of dimension n and admit a modular interpretation deduced from that of the Grassmann varieties. Together, they form a simplicial scheme which "compactifies" the classifying simplicial scheme of PGL_r consisting of all the quotients $\mathrm{PGL}_r^{n+1} / \mathrm{PGL}_r$, $n \geq 0$.

Translated by Mingyi Zhang